ON ACCELERATION OF THE KRASNOSEL'SKIĬ-MANN FIXED POINT ALGORITHM BASED ON CONJUGATE GRADIENT METHOD FOR SMOOTH OPTIMIZATION

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ABSTRACT. This paper considers the problem of finding a fixed point of a nonexpansive mapping on a real Hilbert space and proposes a novel algorithm to accelerate the Krasnosel'skiĭ-Mann algorithm. To this goal, we first consider an unconstrained smooth convex minimization problem, which is an example of a fixed point problem, and show that the Krasnosel'skiĭ-Mann algorithm to solve the minimization problem is based on the steepest descent method. Next, we focus on conjugate gradient methods, which are popular acceleration methods of the steepest descent method, and devise an algorithm blending the conjugate gradient methods with the Krasnosel'skiĭ-Mann algorithm. We prove that, under realistic assumptions, our algorithm converges to a fixed point of a nonexpansive mapping in the sense of the weak topology of a Hilbert space. We perform convergence rate analysis on our algorithm. We numerically compare our algorithm with the Krasnosel'skiĭ-Mann algorithm and show that it reduces the running time and iterations needed to find a fixed point compared with that algorithm.

1. INTRODUCTION

Consider the following *fixed point problem for a nonexpansive mapping* [3, Chapter 4], [7, Chapter 3], [8, Chapter 1], [18, Chapter 3].

Problem 1.1. Let H be a real Hilbert space with norm $\|\cdot\|$, and let $T\colon H \to H$ be nonexpansive, i.e., $\|Tx - Ty\| \leq \|x - y\|$ $(x, y \in H)$. Then,

$$\text{find } x^\star \in \mathrm{F}(T) := \{x^\star \in H : Tx^\star = x^\star\},$$

where one assumes $\mathrm{F}(T) \neq \emptyset$.

A number of iterative algorithms (see, e.g., [1, 2, 13]) have been presented to solve nonlinear problems related to Problem 1.1. A useful algorithm for solving Problem 1.1 is the *Krasnosel'skiĭ-Mann algorithm* [3, Subchapter 5.2], [14, 15] defined by $x_0 \in H$ and

$$x_{n+1} := \alpha_n x_n + (1 - \alpha_n) T x_n \ \ (n \in \mathbb{N}), \tag{1.1}$$

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where $\{\alpha_n\} \subset (0,1)$ satisfies $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$. The Krasnosel'skiĭ-Mann algorithm (1.1) weakly converges to a point in $\mathrm{F}(T)$ [3, Theorem 5.14].

Let us provide an important example of Problem 1.1. Suppose that $f\colon H \to \mathbb{R}$ is convex and Fréchet differentiable and its gradient, denoted by ∇f, is Lipschitz continuous; i.e., there exists $L > 0$ such that $\|\nabla f(x) - \nabla f(y)\| \leq L\|x-y\|$ $(x, y \in H)$. Define $T^f\colon H \to H$ by

$$T^f := I - \alpha \nabla f, \tag{1.2}$$

where I stands for the identity mapping on H and $\alpha \in (0, 2/L]$. Accordingly, T^f satisfies the nonexpansivity condition (see, e.g., [9, Proposition 2.3]) and

$$\mathrm{F}\left(T^f\right) = \left\{x^\star \in H\colon f\left(x^\star\right) = \min_{x \in H} f(x)\right\}.$$

Therefore, we can solve the problem of minimizing f over H by using the Krasnosel'skiĭ-Mann algorithm (1.1) with $T := T^f$, i.e.,

$$\begin{cases} d_{n+1}^f := -\nabla f\left(x_n\right), \\ y_n := T^f\left(x_n\right) = x_n - \alpha \nabla f\left(x_n\right) = x_n + \alpha d_{n+1}^f, \\ x_{n+1} := \alpha_n x_n + \left(1-\alpha_n\right) y_n \quad (n \in \mathbb{N}). \end{cases} \tag{1.3}$$

We can see that algorithm (1.3) uses the *steepest descent direction* [16, Subchapter 3.3], $d_{n+1}^f := -\nabla f(x_n)$, of f at x_n, and hence, algorithm (1.3) is based on the *steepest descent method*.

Here, we focus on the *conjugate gradient methods* [16, Chapter 5] that can find a minimizer of f over H faster than the steepest descent method. The *conjugate gradient direction* of f at x_n $(n \in \mathbb{N})$ can be formulated as follows.

$$d_{n+1}^{f,\mathrm{CGD}} := -\nabla f\left(x_n\right) + \beta_n d_n^{f,\mathrm{CGD}},$$

where $d_0^{f,\mathrm{CGD}} := -\nabla f(x_0)$ and $\{\beta_n\} \subset (0,\infty)$ (See [16, Chapter 5] for the examples of β_n), which, together with (1.2), implies that

$$d_{n+1}^{f,\mathrm{CGD}} = \frac{1}{\alpha}\left(T^f\left(x_n\right) - x_n\right) + \beta_n d_n^{f,\mathrm{CGD}}. \tag{1.4}$$

Therefore, by replacing $d_{n+1}^f := -\nabla f(x_n)$ in algorithm (1.3) with $d_{n+1}^{f,\mathrm{CGD}}$ defined by (1.4), we can formulate the following algorithm for solving Problem 1.1: given $\alpha > 0$, $\{\alpha_n\}, \{\beta_n\} \subset (0,\infty)$, and $x_0, d_0 := (Tx_0 - x_0)/\alpha \in H$,

$$\begin{cases} d_{n+1} := \dfrac{1}{\alpha}\left(Tx_n - x_n\right) + \beta_n d_n, \\ y_n := x_n + \alpha d_{n+1}, \\ x_{n+1} := \alpha_n x_n + \left(1-\alpha_n\right) y_n \quad (n \in \mathbb{N}). \end{cases} \tag{1.5}$$

In this paper, we prove that, under certain assumptions, algorithm (1.5) weakly converges to a point in $\mathrm{F}(T)$. Moreover, we numerically compare algorithm (1.5) with the Krasnosel'skiĭ-Mann algorithm (1.1) and show that it performs better than that algorithm (1.1).

This paper is organized as follows. Section 2 gives the mathematical preliminaries. Section 3 devises the acceleration algorithm for solving Problem 1.1 and presents its convergence analysis. It also presents the convergence rate of the proposed algorithm. Section 4 applies the proposed and conventional algorithms to concrete fixed point problems and provides numerical examples for them. Section 5 concludes the paper.

2. Mathematical Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$, and let $\mathbb{N}$ be the set of all positive integers including zero.

Proposition 2.1. [20, Lemma 1] *Let $\{a_n\}, \{b_n\} \subset (0, \infty)$ be sequences with $a_{n+1} \leq a_n + b_n$ $(n \in \mathbb{N})$. If $\sum_{n=0}^{\infty} b_n < \infty$, $\lim_{n\to\infty} a_n$ exists.*

Proposition 2.2. [19, Subchapter 6.2] *Let $\{a_n\} \subset [0, \infty)$ satisfy $\sum_{n=0}^{\infty} a_n = \infty$ and let $\{b_n\} \subset [0, \infty)$. If $\sum_{n=0}^{\infty} a_n b_n < \infty$, then $\liminf_{n\to\infty} b_n = 0$.*

Proposition 2.3. [17, Lemma 1] *Suppose that $\{x_n\} \subset H$ converges weakly to $x \in H$ and $y \neq x$. Then, $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$.*

Suppose that $C \subset H$ is nonempty, closed, and convex. A mapping, $T \colon C \to C$, is said to be *nonexpansive* [3, Definition 4.1(ii)], [7, (3.2)], [8, Subchapter 1.1], [18, Subchapter 3.1] if $\|Tx - Ty\| \leq \|x - y\|$ $(x, y \in C)$. The *fixed point set* of $T \colon C \to C$ is denoted by $\mathrm{F}(T) := \{x \in C \colon Tx = x\}$. The *metric projection* onto C [3, Subchapter 4.2, Chapter 28] is denoted by P_C. It is defined by $P_C(x) \in C$ and $\|x - P_C(x)\| = \inf_{y\in C} \|x - y\|$ $(x \in H)$. P_C is nonexpansive with $\mathrm{F}(P_C) = C$ [3, Proposition 4.8, (4.8)].

Proposition 2.4. *Suppose that $C \subset H$ is nonempty, closed, and convex, and $T \colon C \to C$ is nonexpansive. Then,*

(i) [3, Corollary 4.15], [7, Lemma 3.4], [8, Proposition 5.3], [18, Theorem 3.1.6] $\mathrm{F}(T)$ *is closed and convex.*
(ii) [3, Theorem 4.19], [7, Theorem 3.1], [8, Theorem 5.1], [18, Corollary 3.1.7] $\mathrm{F}(T)$ *is nonempty if C is bounded.*

3. Acceleration of the Krasnosel'skiĭ-Mann algorithm

Suppose that $T \colon H \to H$ is nonexpansive with $\mathrm{F}(T) \neq \emptyset$. The following algorithm can be used to solve Problem 1.1.

Algorithm 3.1.
Step 0. Choose $\alpha > 0$ and $x_0 \in H$ arbitrarily, and set $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, \infty)$. Compute $d_0 := (Tx_0 - x_0)/\alpha$.
Step 1. Given $x_n, d_n \in H$, compute $d_{n+1} \in H$ by

$$d_{n+1} := \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n.$$

Compute $x_{n+1} \in H$ as follows.

$$\begin{cases} y_n := x_n + \alpha d_{n+1}, \\ x_{n+1} := \alpha_n x_n + (1 - \alpha_n) y_n. \end{cases}$$

Put $n := n + 1$, and go to Step 1.

We can check that Algorithm 3.1 coincides with the Krasnosel'skiĭ-Mann algorithm (1.1) when $\beta_n := 0$ $(n \in \mathbb{N})$.

This section makes the following assumptions.

Assumption 3.1. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy

$$\text{(C1)} \sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = \infty \text{ and (C2)} \sum_{n=0}^{\infty} \beta_n < \infty.$$

Moreover,

(C3) $\mathrm{F}(T)$ is nonempty, and (C4) $\{Tx_n - x_n\}$ is bounded.

Examples of $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying (C1) and (C2) are $\alpha_n := a \in (0, 1)$ $(n \in \mathbb{N})$ and $\beta_n := 1/(n + 1)^b$ $(n \in \mathbb{N})$, where $b > 1$. Suppose that $\mathrm{F}(T)$ is nonempty and bounded. Then, we can set a bounded, closed convex set $C \supset \mathrm{F}(T)$ such that P_C can be computed within a finite number of arithmetic operations (e.g., C is a closed ball with a large enough radius). Hence, we can compute

$$x_{n+1} := P_C \left(\alpha_n x_n + (1 - \alpha_n) y_n\right) \tag{3.1}$$

instead of x_{n+1} in Algorithm 3.1. Since $\{x_n\} \subset C$ and C is bounded, $\{x_n\}$ is bounded. The nonexpansivity of T guarantees that $\|Tx_n - x\| \leq \|x_n - x\|$ $(x \in \mathrm{F}(T))$, which means that $\{Tx_n\}$ is bounded. Accordingly, the boundedness of $\mathrm{F}(T)$ implies (C4). We can prove that Algorithm 3.1 with (3.1) weakly converges to a point in $\mathrm{F}(T)$ by referring to the proof of Theorem 3.1.

Let us consider the case where $\mathrm{F}(T)$ is unbounded. In this case, we cannot choose a bounded C satisfying $\mathrm{F}(T) \subset C$. Although we can execute Algorithm 3.1, we need to verify the boundedness of $\{Tx_n - x_n\}$. Instead, we can apply the Krasnosel'skiĭ-Mann algorithm (1.1) to this case without any problem [3, Theorem 5.14]. That is, when $\mathrm{F}(T)$ is unbounded, we should execute the Krasnosel'skiĭ-Mann algorithm. The trouble is that the Krasnosel'skiĭ-Mann algorithm would converge slowly because it is based on the steepest descent method (see section 1). Hence, in this case, it would be desirable to execute not only the Krasnosel'skiĭ-Mann algorithm but also Algorithm 3.1.

Let us do a convergence analysis of Algorithm 3.1.

Theorem 3.1. *Under Assumption 3.1, the sequence $\{x_n\}$ in Algorithm 3.1 weakly converges to a fixed point of T.*

Theorem 5.5 in [4] when $e_n := \beta_n d_n \quad (n \in \mathbb{N})$ is similar to Theorem 3.1. However, the next subsection shall provide the proof of Theorem 3.1.

3.1. **Proof of Theorem 3.1.** We first show the following.

Lemma 3.1. *Suppose that Assumption 3.1 holds. Then,*

(i) $\{d_n\}$ *is bounded.*
(ii) $\lim_{n\to\infty} \|x_n - u\|$ *exists for all* $u \in \mathrm{F}(T)$. *In particular,* $\{x_n\}$ *is bounded.*
(iii) $\{y_n\}$ *is bounded.*

Proof. (i) Condition (C2) ensures that $\lim_{n\to\infty} \beta_n = 0$. Accordingly, there exists $n_0 \in \mathbb{N}$ such that $\beta_n \leq 1/2$ for all $n \geq n_0$. Put $M_1 := \max\{\|d_{n_0}\|, (2/\alpha)\sup_{n\in\mathbb{N}} \|Tx_n - x_n\|\}$. Condition (C4) implies that $M_1 < \infty$. Assume that $\|d_n\| \leq M_1$ for some $n \geq n_0$. From the triangle inequality, we find that

$$\|d_{n+1}\| = \left\| \frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n \right\| \leq \frac{1}{\alpha} \|Tx_n - x_n\| + \beta_n \|d_n\| \leq M_1.$$

This means that $\|d_n\| \leq M_1$ for all $n \geq n_0$, i.e., $\{d_n\}$ is bounded.

(ii) The definition of y_n $(n \in \mathbb{N})$ implies that

$$\begin{aligned} y_n &= x_n + \alpha \left(\frac{1}{\alpha}(Tx_n - x_n) + \beta_n d_n \right) \\ &= Tx_n + \alpha\beta_n d_n. \end{aligned} \tag{3.2}$$

The triangle inequality and (3.2) mean that, for all $u \in \mathrm{F}(T)$ and for all $n \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n x_n + (1-\alpha_n)(Tx_n + \alpha\beta_n d_n) - u\| \\ &= \|\alpha(x_n - u) + (1-\alpha_n)(Tx_n - u + \alpha\beta_n d_n)\| \\ &\leq \alpha_n \|x_n - u\| + (1-\alpha_n)\|Tx_n - u\| + (1-\alpha_n)\alpha\beta_n \|d_n\|, \end{aligned}$$

which, together with the nonexpansivity of T, $1 - \alpha_n < 1$ $(n \in \mathbb{N})$, and $\|d_n\| \leq M_1$ $(n \geq n_0)$, implies that, for all $u \in \mathrm{F}(T)$ and for all $n \geq n_0$,

$$\|x_{n+1} - u\| \leq \|x_n - u\| + \alpha M_1 \beta_n.$$

Proposition 2.1 and (C2) guarantee that $\lim_{n\to\infty} \|x_n - u\|$ exists for all $u \in \mathrm{F}(T)$. This means $\{x_n\}$ is bounded.

(iii) The definition of y_n $(n \in \mathbb{N})$ and the boundedness of $\{x_n\}$ and $\{d_n\}$ imply that $\{y_n\}$ is also bounded. This completes the proof. □

Lemma 3.2. *Suppose that Assumption 3.1 holds. Then,*

(i) $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.
(ii) *There exists a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *which weakly converges to a fixed point of* T.

Proof. (i) Choose $u \in \mathrm{F}(T)$ arbitrarily. From the equality, $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ $(\alpha \in [0,1], x, y \in H)$ [19, Theorem

6.1.2], we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - u\|^2 &= \|\alpha_n x_n + (1-\alpha_n) y_n - u\|^2 \\ &= \|\alpha_n (x_n - u) + (1-\alpha_n)(y_n - u)\|^2 \\ &= \alpha_n \|x_n - u\|^2 + (1-\alpha_n)\|y_n - u\|^2 - \alpha_n(1-\alpha_n)\|x_n - y_n\|^2.\end{aligned}$$

From (3.2), the nonexpansivity of T, and the inequality, $\|x+y\|^2 \leq \|x\|^2 + 2\langle x+y, y\rangle$ $(x, y \in H)$, we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|y_n - u\|^2 &= \|(Tx_n - u) + \alpha\beta_n d_n\|^2 \\ &\leq \|Tx_n - u\|^2 + 2\alpha\beta_n \langle y_n - u, d_n\rangle \\ &\leq \|x_n - u\|^2 + M_2\beta_n,\end{aligned}$$

where $M_2 := \sup_{n\in\mathbb{N}} 2\alpha|\langle y_n - u, d_n\rangle| < \infty$. Hence, from $\|x_n - y_n\| = \alpha\|d_{n+1}\|$ $(n \in \mathbb{N})$, we find that, for all $n \in \mathbb{N}$,

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 + M_2\beta_n - \alpha^2\alpha_n(1-\alpha_n)\|d_{n+1}\|^2.$$

Therefore, for all $n \in \mathbb{N}$,

$$\alpha^2\alpha_n(1-\alpha_n)\|d_{n+1}\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + M_2\beta_n.$$

Summing up this inequality from $n = 0$ to $n = N \in \mathbb{N}$ yields

$$\begin{aligned}\alpha^2 \sum_{n=0}^{N} \alpha_n(1-\alpha_n)\|d_{n+1}\|^2 &\leq \|x_0 - u\|^2 - \|x_{N+1} - u\|^2 + M_2 \sum_{n=0}^{N} \beta_n \\ &\leq \|x_0 - u\|^2 + M_2 \sum_{n=0}^{\infty} \beta_n. \qquad (3.3)\end{aligned}$$

Accordingly, (C2) guarantees that

$$\alpha^2 \sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\|d_{n+1}\|^2 < \infty.$$

Hence, Proposition 2.2 and (C1) mean that

$$\liminf_{n\to\infty} \|d_{n+1}\| = 0. \qquad (3.4)$$

From the definition of d_{n+1} $(n \in \mathbb{N})$, we have that, for all $n \geq n_0$,

$$\frac{1}{\alpha}\|Tx_n - x_n\| \leq \|d_{n+1}\| + \beta_n\|d_n\| \leq \|d_{n+1}\| + M_1\beta_n, \qquad (3.5)$$

which, together with (3.4) and $\lim_{n\to\infty} \beta_n = 0$, implies that

$$\begin{aligned}\frac{1}{\alpha}\liminf_{n\to\infty} \|Tx_n - x_n\| &\leq \liminf_{n\to\infty} (\|d_{n+1}\| + M_1\beta_n) \\ &= \liminf_{n\to\infty} \|d_{n+1}\| + M_1 \lim_{n\to\infty} \beta_n \\ &= 0.\end{aligned}$$

Thus, we find that

$$\liminf_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.6}$$

From (3.2) and the triangle inequality, we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &= \|Tx_{n+1} - \alpha_n x_n - (1-\alpha_n)(Tx_n + \alpha\beta_n d_n)\| \\ &\leq \alpha_n \|Tx_{n+1} - x_n\| + (1-\alpha_n)\|Tx_{n+1} - Tx_n\| \\ &\quad + \alpha(1-\alpha_n)\beta_n \|d_n\|,\end{aligned}$$

which, together with $\|d_n\| \leq M_1$ $(n \geq n_0)$, the nonexpasivity of T, and the triangle inequality, implies that, for all $n \geq n_0$,

$$\begin{aligned}\|Tx_{n+1} - x_{n+1}\| &\leq \alpha_n \|Tx_{n+1} - x_n\| + (1-\alpha_n)\|x_{n+1} - x_n\| + \alpha M_1 (1-\alpha_n)\beta_n \\ &\leq \alpha_n \|Tx_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \alpha M_1 (1-\alpha_n)\beta_n.\end{aligned}$$

Hence, we find that, for all $n \geq n_0$,

$$\begin{aligned}(1-\alpha_n)\|Tx_{n+1} - x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \alpha M_1 (1-\alpha_n)\beta_n \\ &= \|\alpha_n x_n + (1-\alpha_n)(Tx_n + \alpha\beta_n d_n) - x_n\| \\ &\quad + \alpha M_1 (1-\alpha_n)\beta_n \\ &= (1-\alpha_n)\|Tx_n - x_n + \alpha\beta_n d_n\| + \alpha M_1 (1-\alpha_n)\beta_n \\ &\leq (1-\alpha_n)\|Tx_n - x_n\| + 2\alpha M_1 (1-\alpha_n)\beta_n,\end{aligned}$$

which means that, for all $n \geq n_0$,

$$\|Tx_{n+1} - x_{n+1}\| \leq \|Tx_n - x_n\| + 2\alpha M_1 \beta_n. \tag{3.7}$$

Therefore, Proposition 2.1 and (C2) guarantee the existence of $\lim_{n\to\infty} \|Tx_n - x_n\|$. Equation (3.6) leads us to

$$\lim_{n\to\infty} \|Tx_n - x_n\| = \liminf_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{3.8}$$

(ii) Since $\{x_n\}$ is bounded, there exists $\{x_{n_i}\} \subset \{x_n\}$ which weakly converges to $z \in H$. Assume that $z \notin \mathrm{F}(T)$, i.e., $z \neq Tz$. Proposition 2.3, (3.8), and the nonexpansivity of T ensure that

$$\begin{aligned}\liminf_{n\to\infty} \|x_n - z\| &< \liminf_{n\to\infty} \|x_n - Tz\| \\ &= \liminf_{n\to\infty} \|x_n - Tx_n + Tx_n - Tz\| \\ &= \liminf_{n\to\infty} \|Tx_n - Tz\| \\ &\leq \liminf_{n\to\infty} \|x_n - z\|.\end{aligned}$$

This is a contradiction. Hence, $z \in \mathrm{F}(T)$. This completes the proof. □

Now, we are in the position to prove Theorem 3.1.

Proof. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$. The boundedness of $\{x_n\}$ implies that $\{x_{n_j}\}$ weakly converges to $w \in H$. A similar discussion as in the proof of Lemma 3.2(ii) leads us to $w \in \mathrm{F}(T)$.

Assume that $z \neq w$. Then, Lemma 3.1(ii) and Proposition 2.3 mean that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z\| &= \lim_{i\to\infty} \|x_{n_i} - z\| < \lim_{i\to\infty} \|x_{n_i} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\| = \lim_{j\to\infty} \|x_{n_j} - w\| \\ &< \lim_{j\to\infty} \|x_{n_j} - z\| = \lim_{n\to\infty} \|x_n - z\|.\end{aligned}$$

This is a contradiction. Hence, $z = w$. This guarantees that $\{x_n\}$ weakly converges to a fixed point of T. This completes the proof. □

Let us examine the convergence rate of Algorithm 3.1.

Theorem 3.2. *Suppose that Assumption 3.1 holds and the sequence $\{\|d_n\|\}$ is monotone decreasing, i.e. $\|d_{n+1}\| \leq \|d_n\|$ for all $n \in \mathbb{N}$. Then,*

$$\|Tx_{N+1} - x_{N+1}\| \leq \sqrt{\frac{d(x_0, \mathrm{F}(T))^2 + M_2 \sum_{n=0}^{\infty} \beta_n}{\sum_{n=0}^{N} \alpha_n (1-\alpha_n)}} + \alpha M_1 \beta_N.$$

for all $N \in \mathbb{N}$, where $d(x_0, \mathrm{F}(T)) := \inf_{u \in \mathrm{F}(T)} \|x_0 - u\|$.

Proof. Fix $u \in \mathrm{F}(T)$ and $N \in \mathbb{N}$ arbitrarily. From (3.3), we have

$$\alpha^2 \sum_{n=0}^{N} \alpha_n (1-\alpha_n) \|d_{n+1}\|^2 \leq \|x_0 - u\|^2 + M_2 \sum_{n=0}^{\infty} \beta_n.$$

Accordingly, the monotone decreasing property of $\{\|d_n\|\}$ ensures that

$$\alpha^2 \|d_{N+1}\|^2 \sum_{n=0}^{N} \alpha_n (1-\alpha_n) \leq \|x_0 - u\|^2 + M_2 \sum_{n=0}^{\infty} \beta_n.$$

Thus, dividing both sides of the above inequality by $\alpha^2 \sum_{n=0}^{N} \alpha_n (1-\alpha_n) > 0$ yields

$$\|d_{N+1}\|^2 \leq \frac{\|x_0 - u\|^2 + M_2 \sum_{n=0}^{\infty} \beta_n}{\alpha^2 \sum_{n=0}^{N} \alpha_n (1-\alpha_n)}.$$

From (3.5), we find

$$\begin{aligned}\|Tx_N - x_N\| &\leq \alpha \|d_{N+1}\| + \alpha M_1 \beta_N \\ &\leq \sqrt{\frac{\|x_0 - u\|^2 + M_2 \sum_{n=0}^{\infty} \beta_n}{\sum_{n=0}^{N} \alpha_n (1-\alpha_n)}} + \alpha M_1 \beta_N.\end{aligned}$$

This completes the proof. □

Remark 3.1. The Krasnosel'skiĭ-Mann algorithm (1.1) has the following convergence rate [6, Subsection 3.2]:

$$\|Tx_{N+1} - x_{N+1}\| \leq \sqrt{\frac{d(x_0, \mathrm{F}(T))^2}{\sum_{n=1}^{N} \alpha_n (1-\alpha_n)}}$$

for all $u \in \mathrm{F}(T)$ and $N \in \mathbb{N}$.

4. Numerical Examples

The theoretical convergence rate analysis (Theorem 3.2) did not explicitly show the superiority of Algorithm 3.1 over the Krasnosel'skiĭ-Mann algorithm (1.1). Therefore, let us apply the Krasnosel'skiĭ-Mann algorithm (1.1) and Algorithm 3.1 to the following problem [5, section I, Framework 2], [10, Problem 10], [11, Problem 2.1], [12, Subsection 4.2], [21, Definition 4.1].

Problem 4.1. Suppose that $C_0 \subset \mathbb{R}^N$ is a nonempty, bounded, closed convex set, $C_i \subset \mathbb{R}^N$ $(i = 1, 2, \ldots, m)$ is a nonempty, closed convex set and $\Phi(x)$ is the mean square value of the distances from $x \in \mathbb{R}^N$ to C_i $(i = 1, 2, \ldots, m)$, i.e.,

$$\Phi(x) := \frac{1}{m}\sum_{i=1}^{m} d\left(x, C_i\right)^2 = \frac{1}{m}\sum_{i=1}^{m}\left(\min_{y \in C_i}\|x - y\|\right)^2 \quad (x \in \mathbb{R}^N).$$

Then,

$$\text{find } x^\star \in C_\Phi := \left\{x^\star \in C_0 \colon \Phi\left(x^\star\right) = \min_{y \in C_0}\Phi(y)\right\}.$$

The set C_Φ is called the *generalized convex feasible set* [5, section I, Framework 2], [21, Definition 4.1] and is a subset of C_0 whose elements are closest to C_is in the sense of the mean square norm. The set C_Φ is well-defined even if $\bigcap_{i=0}^{m} C_i = \emptyset$. This is because it is the set of all minimizers of Φ over C_0. The boundedness and closedness of C_0 guarantee $C_\Phi \neq \emptyset$ [21, Remark 4.3(a)]. Moreover, the condition $C_\Phi = \bigcap_{i=0}^{m} C_i$ holds when $\bigcap_{i=0}^{m} C_i \neq \emptyset$, which means C_Φ is a generalization of $\bigcap_{i=0}^{m} C_i$.

Here, we can define a mapping $T \colon \mathbb{R}^N \to \mathbb{R}^N$ by

$$T := P_0\left(\frac{1}{m}\sum_{i=1}^{m} P_i\right), \tag{4.1}$$

where $P_i := P_{C_i}$ $(i = 0, 1, \ldots, m)$ stands for the metric projection onto C_i. Accordingly, Proposition 4.2 in [21] guarantees that T defined by (4.1) is nonexpansive and

$$\mathrm{F}(T) = C_\Phi.$$

Therefore, Problem 4.1 coincides with Problem 1.1 with T defined by (4.1).

The experiment used an Apple MacBook Air with a 1.30GHz Intel(R) Core(TM) i5-4250U CPU and 4GB DDR3 memory. The Krasnosel'skiĭ-Mann algorithm (1.1) and Algorithm 3.1 were written in C and complied using clang-425.0.28. The operating system of the computer was Mac OSX version 10.8.5.

We set $\alpha := 1$, $\alpha_n := 1/2$ $(n \in \mathbb{N})$, and $\beta_n := 1/(n+1)^{1.001}$ $(n \in \mathbb{N})$. In the experiment, we set C_i $(i = 0, 1, \ldots, m)$ as a closed ball with center

$c_i \in \mathbb{R}^N$ and radius $r_i > 0$. Thus, P_i $(i = 0, 1, \ldots, m)$ can be computed with

$$P_i(x) := x + \frac{\|c_i - x\| - r_i}{\|c_i - x\|}(c_i - x) \text{ if } \|c_i - x\| > r_i,$$

or $P_i(x) := x$ if $\|c_i - x\| \leq r_i$.

We also set $N := 100$, $m := 3$, $r_i := 1$ $(i = 0, 1, 2, 3)$, and $c_0 := 0$. The experiment used random vectors $c_i \in \mathbb{R}^{100}$ $(i = 1, 2, 3)$. The vectors c_i $(i = 1, 2, 3)$ were generated using the function `random` based on the 54-bit version of L'Ecuyer's MRG32k3a algorithm in Racket v6.0.

In the experiment, we performed 100 samplings, each starting from different random initial points in $(-16, 16)^N$. The 100 initial points were generated with the function `gsl_rng_uniform` based on the MT19937 algorithm in the GNU Scientific Library 1.14.91. We averaged the results of the 100 samplings.

We set $c_i \in (-N^{-\frac{1}{2}}, N^{-\frac{1}{2}})^N$ $(i = 1, 2, 3)$ in order to consider the case where $\bigcap_{i=0}^{3} C_i \neq \emptyset$. We also set $c_i \in \{(-10, -2) \cup (2, 10)\}^N$ $(i = 1, 2, 3)$ to consider the case where $\bigcap_{i=0}^{3} C_i = \emptyset$. Figure 1 describes the behaviors of $\|Tx_n - x_n\|$ for the Krasnosel'skiĭ-Mann algorithm (1.1) and Algorithm 3.1 (Proposed). The x-axis and y-axis represent the elapsed time and mean value of $\|Tx_n - x_n\|$.

Figure 1(a) is for $\bigcap_{i=0}^{3} C_i \neq \emptyset$. It shows that Algorithm 3.1 dramatically reduces the time required to satisfy $\|Tx_n - x_n\| < 10^{-6}$ compared with the Krasnosel'skiĭ-Mann algorithm (1.1). We found that the Krasnosel'skiĭ-Mann algorithm (1.1) took 47 iterations to satisfy $\|Tx_n - x_n\| < 10^{-6}$, whereas Algorithm 3.1 took only eight.

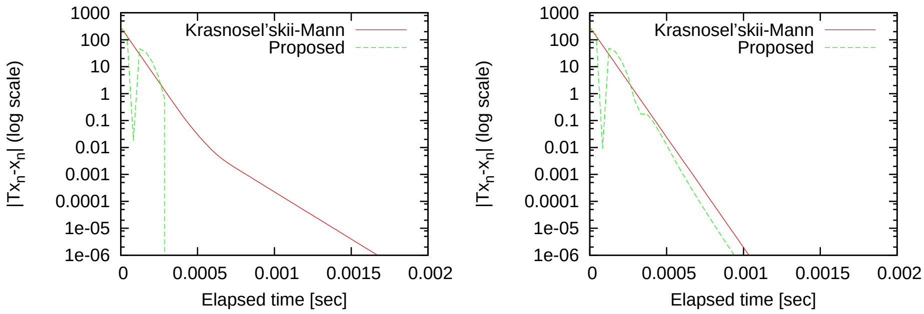
Figure 1(b) is $\bigcap_{i=0}^{3} C_i = \emptyset$. We can see that Algorithm 3.1 converges to a point in $C_\Phi = \mathrm{F}(T)$ faster than the Krasnosel'skiĭ-Mann algorithm (1.1). The Krasnosel'skiĭ-Mann algorithm (1.1) took 29 iterations to satisfy $\|Tx_n - x_n\| < 10^{-6}$, whereas ours took 23.

The Krasnosel'skiĭ-Mann algorithm (1.1) satisfies $\|Tx_{n+1} - x_{n+1}\| \leq \|Tx_n - x_n\|$ $(n \in \mathbb{N})$ (see, e.g., [3, (5.14)]). This is also verified in Figure 1. Meanwhile, (3.7) guarantees that Algorithm 3.1 satisfies

$$\|Tx_{n+1} - x_{n+1}\| \leq \|Tx_n - x_n\| + \frac{2M_1}{(n+1)^{1.001}} \ (n \in \mathbb{N}).$$

This implies that $\{Tx_n - x_n\}$ in Algorithm 3.1 does not monotonically decrease. However, for large enough n, $2M_1/(n+1)^{1.001} \approx 0$. Therefore, we can see that $\{Tx_n - x_n\}$ will monotonically decrease for large enough n. Such a trend is visible in Figure 1.

From the above discussion, we can conclude that Algorithm 3.1 performs better than the Krasnosel'skiĭ-Mann algorithm in terms of the number of iteration and elapsed time to reach a solution.



(a) Case of $\bigcap_{i=0}^{3} C_i \neq \emptyset$ (b) Case of $\bigcap_{i=0}^{3} C_i = \emptyset$

FIGURE 1. Behavior of $\|Tx_n - x_n\|$ for the Krasnosel'skiĭ-Mann algorithm and Algorithm 3.1 (Proposed)

5. Conclusion and Future Work

This paper presented an algorithm to accelerate the Krasnosel'skiĭ-Mann algorithm for finding a fixed point of a nonexpansive mapping on a real Hilbert space and its convergence analysis. The convergence analysis guarantees that the proposed algorithm weakly converges to a fixed point of a nonexpansive mapping under certain assumptions. We numerically compared the abilities of the proposed and Krasnosel'skiĭ-Mann algorithms on concrete fixed point problems. The results showed that the proposed algorithm performs better than the Krasnosel'skiĭ-Mann algorithm.

The numerical results also indicated that the proposed algorithm is unstable in the early stages of operation. Therefore, in the future, we should develop acceleration algorithms that behave stably in their search for a fixed point of a nonexpansive mapping.

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