PARALLEL COMPUTING PROXIMAL METHOD FOR NONSMOOTH CONVEX OPTIMIZATION WITH FIXED POINT CONSTRAINTS OF QUASI-NONEXPANSIVE MAPPINGS

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Abstract. We present a parallel computing proximal method for solving the problem of minimizing the sum of convex functions over the intersection of fixed point sets of quasi-nonexpansive mappings in a real Hilbert space. We also provide a convergence analysis of the method for constant and diminishing step sizes under certain assumptions as well as a convergence-rate analysis for a diminishing step size. Numerical comparisons show that the performance of the algorithm is comparable with existing subgradient methods.

Keywords. Fixed point; Nonsmooth convex optimization; Parallel computing; Proximal method; Quasi-nonexpansive mapping.

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1. INTRODUCTION

In this paper, we consider the following problem [7, Problem 2.1] (see [3, 9, 10] for applications of Problem 1.1):

Problem 1.1. Let H be a real Hilbert space. Suppose that

(A1) $Q_i \colon H \to H$ $(i \in \mathscr{I} := \{1, 2, \ldots, I\})$ is quasi-firmly nonexpansive;

(A2) $f_i \colon H \to \mathbb{R}$ $(i \in \mathscr{I})$ is convex and continuous with $\mathrm{dom}(f_i) := \{x \in H \colon f_i(x) < +\infty\} = H$.

Then,

$$\text{minimize } f(x) := \sum_{i \in \mathscr{I}} f_i(x) \text{ subject to } x \in X := \bigcap_{i \in \mathscr{I}} \mathrm{Fix}(Q_i),$$

where one assumes that there exists a solution of Problem 1.1 (see Sections 2 and 4 for the details).

Algorithms for solving this problem have been proposed in [7, 9]. Reference [7] proposed parallel and incremental subgradient methods for solving Problem 1.1 and provided convergence as well as convergence-rate analyses. Reference [9, 10] proposed stochastic fixed point optimization algorithms for solving a convex stochastic optimization problem that is to minimize the expectation of f_is over $\mathrm{Fix}(Q_1)$. The stochastic fixed point optimization algorithms can be applied to the classifier ensemble problem.

There are methods for solving Problem 1.1 where Q_i is taken to be a nonexpansive mapping, which is a stronger assumption than a quasi-nonexpansive mapping. Subgradient methods were presented in [4, 5, 6, 11], while proximal methods were presented in [8, 16].

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In this paper, we present a parallel method for solving Problem 1.1. The method is obtained by combining the parallel method in [7] with the proximal method in [8]. We also present a convergence analysis for a constant step size and a diminishing step size. The analysis shows that the proposed method with a small constant step size may approximate a solution to Problem 1.1 (Theorem 3.1) and that with a diminishing step size it converges to a solution under certain assumptions (Theorem 3.2). We also provide a convergence-rate analysis with a diminishing step size (Theorem 3.3). Finally, we numerically compare the proposed method with the existing subgradient methods.

This paper is organized as follows. Section 2 gives the mathematical preliminaries. Section 3 presents the parallel proximal method for solving Problem 1.1 and analyzes its convergence. Section 4 numerically compares the behaviors of the proposed method and the existing ones. Section 5 concludes the paper with a brief summary.

2. Mathematical Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. We use the standard notation $\mathbb{N}$ for the natural numbers including zero and $\mathbb{R}^N$ for the N-dimensional Euclidean space.

2.1. Quasi-nonexpansivity and demiclosedness.

The fixed point set of a mapping $Q\colon H \to H$ is denoted by

$$\mathrm{Fix}(Q) := \{x \in H \colon Q(x) = x\}.$$

Q is said to be *quasi-nonexpansive* [2, Definition 4.1(iii)] if $\|Q(x) - y\| \leq \|x - y\|$ for all $x \in H$ and for all $y \in \mathrm{Fix}(Q)$. When a quasi-nonexpansive mapping has one fixed point, its fixed point set is closed and convex [2, Proposition 2.6]. Q is said to be *quasi-firmly nonexpansive* [1, Section 3] if, for all $x \in H$ and for all $y \in \mathrm{Fix}(Q)$,

$$\|Q(x) - y\|^2 + \|(\mathrm{Id} - Q)(x)\|^2 \leq \|x - y\|^2,$$

where $\mathrm{Id}(x) := x$ $(x \in H)$. Any quasi-firmly nonexpansive mapping satisfies the quasi-nonexpansivity condition. Moreover, Q is quasi-firmly nonexpansive if and only if $R := 2Q - \mathrm{Id}$ is quasi-nonexpansive [2, Proposition 4.2], which implies that $(1/2)(\mathrm{Id} + R)$ is quasi-firmly nonexpansive when R is quasi-nonexpansive. Let $x, u \in H$ and $(x_n)_{n \in \mathbb{N}} \subset H$. $\mathrm{Id} - Q$ is said to be *demiclosed* if a weak convergence of (x_n) to x and $\lim_{n \to +\infty} \|x_n - Q(x_n) - u\| = 0$ imply $x - Q(x) = u$. $\mathrm{Id} - Q$ is demiclosed when Q is nonexpansive, i.e., $\|Q(x) - Q(y)\| \leq \|x - y\|$ $(x, y \in H)$ [2, Theorem 4.17]. The *metric projection* P_C onto a nonempty, closed convex subset C of H is firmly nonexpansive, i.e., $\|P_C(x) - P_C(y)\|^2 + \|(\mathrm{Id} - P_C)(x) - (\mathrm{Id} - P_C)(y)\|^2 \leq \|x - y\|^2$ $(x, y \in H)$. Moreover, $\mathrm{Fix}(P_C) = C$ [2, Proposition 4.8, (4.8)].

2.2. Convexity, proximal point, and subdifferentiability.

A function $f\colon H \to \mathbb{R}$ is said to be convex if, for all $x, y \in H$ and for all $\alpha \in [0,1]$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$. A function f is said to be *strictly convex* [2, Definition 8.6] if, for all $x, y \in H$ and for all $\alpha \in (0,1)$, $x \neq y$ implies $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$. f is strongly convex with constant β [2, Definition 10.5] if there exists $\beta > 0$ such that, for all $x, y \in H$ and for all $\alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) + (\beta \alpha (1-\alpha)/2)\|x - y\|^2 \leq \alpha f(x) + (1-\alpha) f(y)$.

Let $f\colon H \to (-\infty, +\infty]$ be proper, lower semicontinuous, and convex. Then, the *proximity operator* of f [2, Definition 12.23], [14], denoted by Prox_f, maps every $x \in H$ to the unique minimizer of $f(\cdot) +$

$(1/2)\|x-\cdot\|^2$; i.e.,

$$\{\mathrm{Prox}_f(x)\} = \operatorname*{argmin}_{y\in H}\left[f(y)+\frac{1}{2}\|x-y\|^2\right] \ (x\in H).$$

The uniqueness and existence of $\mathrm{Prox}_f(x)$ are guaranteed for all $x\in H$ [2, Definition 12.23], [13]. We call $\mathrm{Prox}_f(x)$ the *proximal point* of f at x. Let $\mathrm{dom}(f) := \{x\in H\colon f(x) < +\infty\}$ be the domain of a function $f\colon H\to(-\infty,+\infty]$.

The *subdifferential* [2, Definition 16.1] of f is defined by

$$\partial f(x) := \{u\in H\colon f(y)\geq f(x)+\langle y-x,u\rangle \ (y\in H)\} \ (x\in H).$$

We call $u\in\partial f(x)$ the *subgradient* of f at x.

Proposition 2.1. [2, Propositions 12.26, 12.27, 12.28, and 16.14] *Let $f\colon H\to(-\infty,\infty]$ be proper, lower semicontinuous, and convex. Then, the following hold:*

(i) *Let $x,p\in H$. $p=\mathrm{Prox}_f(x)$ if and only if $x-p\in\partial f(p)$ (i.e., $\langle y-p,x-p\rangle+f(p)\leq f(y)$ for all $y\in H$).*
(ii) *Prox_f is firmly nonexpansive with $\mathrm{Fix}(\mathrm{Prox}_f)=\mathrm{argmin}_{x\in H} f(x)$.*
(iii) *If f is continuous at $x\in\mathrm{dom}(f)$, $\partial f(x)$ is nonempty. Moreover, $\delta>0$ exists such that $\partial f(B(x;\delta))$ is bounded, where $B(x;\delta)$ stands for a closed ball with center x and radius δ.*

The following propositions will be used to prove the main theorems in this paper.

Proposition 2.2. [15, Lemma 3.1] *Suppose that $(x_n)_{n\in\mathbb{N}}\subset H$ weakly converges to $\hat{x}\in H$ and $\bar{x}\neq\hat{x}$. Then,* $\liminf_{n\to+\infty}\|x_n-\hat{x}\| < \liminf_{n\to+\infty}\|x_n-\bar{x}\|$.

Proposition 2.3. [2, Theorem 9.1] *When $f\colon H\to\mathbb{R}$ is convex, f is weakly lower semicontinuous if and only if f is lower semicontinuous.*

Proposition 2.4. [12, Lemma 2.1] *Let $(\Gamma_n)_{n\in\mathbb{N}}\subset\mathbb{R}$ and suppose that $(\Gamma_{n_j})_{j\in\mathbb{N}}$ ($\subset(\Gamma_n)_{n\in\mathbb{N}}$) exists such that $\Gamma_{n_j}<\Gamma_{n_j+1}$ for all $j\in\mathbb{N}$. Define $(\tau(n))_{n\geq n_0}\subset\mathbb{N}$ by $\tau(n):=\max\{k\leq n\colon \Gamma_k<\Gamma_{k+1}\}$ for some $n_0\in\mathbb{N}$. Then, $(\tau(n))_{n\geq n_0}$ is increasing and $\lim_{n\to\infty}\tau(n)=+\infty$. Moreover, $\Gamma_{\tau(n)}\leq\Gamma_{\tau(n)+1}$ and $\Gamma_n\leq\Gamma_{\tau(n)+1}$ for all $n\geq n_0$.*

3. Proposed Parallel Proximal Method

Algorithm 1 is the proposed algorithm for solving Problem 1.1.

Let us consider a network system with I users and assume that user i has its own private objective function f_i and mapping Q_i and tries to minimize f_i over $\mathrm{Fix}(Q_i)$. Moreover, let us assume that each user can communicate with other users. Then, at iteration n, each user can have x_n in common. Since user i has its own objective function f_i, it computes $y_{n,i} := \mathrm{Prox}_{\gamma_n f_i}(x_n)$. Moreover, user i has its own constraint set $\mathrm{Fix}(Q_i)$, with which it tries to find a fixed point of Q_i by using $x_{n,i} := Q_i(y_{n,i})$. Since the users can communicate with each other, user i can receive all $x_{n,i}$, and hence, user i can compute $x_{n+1} := (1/I)\sum_{i\in\mathscr{I}} x_{n,i}$.

Algorithm 1 Parallel Proximal Method for solving Problem 1.1

Require: $(\gamma_n)_{n\in\mathbb{N}} \subset (0,+\infty)$

$n \leftarrow 0$, $x_0 \in H$
loop
  for $i = 1$ to $i = I$ **do**
    $x_{n,i} := Q_i(\mathrm{Prox}_{\gamma_n f_i}(x_n))$
  end for
  $x_{n+1} := \dfrac{1}{I}\sum_{i\in\mathscr{I}} x_{n,i}$
  $n \leftarrow n+1$
end loop

Let us compare Algorithm 1 with the existing parallel subgradient method [7, Algorithm 3.1] for solving Problem 1.1. The parallel subgradient method [7, Algorithm 3.1] is as follows:

$$\begin{aligned} &Q_{\alpha,i} := \alpha\mathrm{Id} + (1-\alpha)Q_i,\\ &g_{n,i} \in \partial f_i\left(Q_{\alpha,i}(x_n)\right),\\ &x_{n,i} := Q_{\alpha,i}(x_n) - \lambda_n g_{n,i},\\ &x_{n+1} := \frac{1}{I}\sum_{i\in\mathscr{I}} x_{n,i}. \end{aligned} \tag{3.1}$$

The difference between Algorithms 1 and (3.1) is the form of $x_{n,i}$, i.e., Algorithm 1 uses $x_{n,i} = Q_i(\mathrm{Prox}_{\gamma_n f_i}(x_n))$, while algorithm (3.1) uses $x_{n,i} := Q_{\alpha,i}(x_n) - \lambda_n g_{n,i}$. Section 4 compares the behaviors of Algorithm 1 and algorithm (3.1) for concrete optimization problems.

First, we prove the following lemma:

Lemma 3.1. *Suppose that (A1) and (A2) hold and define* $y_{n,i} := \mathrm{Prox}_{\gamma_n f_i}(x_n)$ *for all* $i \in \mathscr{I}$ *and for all* $n \in \mathbb{N}$. *Then, Algorithm 1 satisfies that, for all* $x \in X$ *and for all* $n \in \mathbb{N}$,

$$\|x_{n+1}-x\|^2 \leq \|x_n - x\|^2 - \frac{1}{I}\sum_{i\in\mathscr{I}}\left\{\|x_n - y_{n,i}\|^2 + \|x_{n,i}-y_{n,i}\|^2\right\} + \frac{2}{I}\gamma_n\sum_{i\in\mathscr{I}}\left(f_i(x) - f_i(y_{n,i})\right).$$

Proof. Let $x \in X$ and $n \in \mathbb{N}$ be fixed arbitrarily. The definition of $y_{n,i} := \mathrm{Prox}_{\gamma_n f_i}(x_n)$ and Proposition 2.1(i) ensure that, for all $i \in \mathscr{I}$,

$$\langle x - y_{n,i}, x_n - y_{n,i}\rangle \leq \gamma_n\left(f_i(x) - f_i(y_{n,i})\right),$$

which, together with $2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2$ $(x,y \in H)$, implies that

$$2\gamma_n\left(f_i(x) - f_i(y_{n,i})\right) \geq \|x-y_{n,i}\|^2 + \|x_n - y_{n,i}\|^2 - \|x - x_n\|^2.$$

Accordingly, for all $i \in \mathscr{I}$,

$$\|y_{n,i}-x\|^2 \leq \|x_n - x\|^2 - \|x_n - y_{n,i}\|^2 + 2\gamma_n\left(f_i(x) - f_i(y_{n,i})\right). \tag{3.2}$$

The definition of $x_{n,i} := Q_i(y_{n,i})$ and (A1) guarantee that, for all $i \in \mathscr{I}$,

$$\|x_{n,i}-x\|^2 \leq \|y_{n,i}-x\|^2 - \|x_{n,i}-y_{n,i}\|^2. \tag{3.3}$$

Hence, (3.2) and (3.3) imply that

$$\|x_{n,i} - x\|^2 \leq \|x_n - x\|^2 - \|x_n - y_{n,i}\|^2 - \|x_{n,i} - y_{n,i}\|^2 + 2\gamma_n (f_i(x) - f_i(y_{n,i})).$$

Summing the above inequality from $i = 1$ to $i = I$ and the convexity of $\|\cdot\|^2$ ensure that

$$\begin{aligned} I\|x_{n+1} - x\|^2 &\leq \sum_{i \in \mathscr{I}} \|x_{n,i} - x\|^2 \\ &\leq I\|x_n - x\|^2 - \sum_{i \in \mathscr{I}} \left\{ \|x_n - y_{n,i}\|^2 + \|x_{n,i} - y_{n,i}\|^2 \right\} + 2\gamma_n \sum_{i \in \mathscr{I}} (f_i(x) - f_i(y_{n,i})), \end{aligned}$$

which completes the proof. □

The convergence analysis of Algorithm 1 depends on the following:

Assumpion 3.1. The sequence $(y_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ is bounded.

Assume that, for all $i \in \mathscr{I}$, $\operatorname{argmin}_{x \in H} f_i(x) (= \mathrm{Fix}(\mathrm{Prox}_{f_i})) \neq \emptyset$ and $\mathrm{Fix}(Q_i)$ is bounded. Then, we can choose in advance of running the algorithm a bounded, closed convex set C_i (e.g., C_i is a closed ball with a large enough radius) satisfying $C_i \supset \mathrm{Fix}(Q_i)$. Accordingly, we can compute

$$x_{n,i} := P_{C_i}[Q_i(y_{n,i})] \in C_i \tag{3.4}$$

instead of $x_{n,i}$ in Algorithm 1. The boundedness of C_i $(i \in \mathscr{I})$ implies that $(x_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ is bounded. Accordingly, $(x_n)_{n \in \mathbb{N}}$ is also bounded. Moreover, Proposition 2.1(ii) ensures that, for all $i \in \mathscr{I}$, for all $n \in \mathbb{N}$, and for all $x \in \mathrm{Fix}(\mathrm{Prox}_{f_i})$, $\|y_{n,i} - x\| \leq \|x_n - x\|$. Hence, the boundedness of $(x_n)_{n \in \mathbb{N}}$ guarantees that $(y_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ is bounded. Hence, it can be assumed that $(x_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ in Algorithm 1 is as in (3.4) in place of Assumption 3.1.

We also have the following lemma:

Lemma 3.2. *Suppose that (A1), (A2), and Assumption 3.1 hold. Then, $(x_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ and $(x_n)_{n \in \mathbb{N}}$ are bounded.*

Proof. Assumption (A1) ensures that, for all $x \in X$, for all $i \in \mathscr{I}$, and for all $n \in \mathbb{N}$,

$$\|x_{n,i} - x\| \leq \|y_{n,i} - x\|,$$

which, together with Assumption 3.1, implies that $(x_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ is bounded. Hence, the definition of x_n implies that $(x_n)_{n \in \mathbb{N}}$ is also bounded. □

3.1. **Constant step-size rule.** The following is a convergence analysis of Algorithm 1 with a constant step size, which indicates that Algorithm 1 with a small constant step size may approximate a solution of Problem 1.1.

Theorem 3.1. *Suppose that (A1), (A2), and Assumption 3.1 hold. Then, Algorithm 1 with $\gamma_n := \gamma > 0$ satisfies that*

$$\liminf_{n \to +\infty} \sum_{i \in \mathscr{I}} \|y_{n,i} - Q_i(y_{n,i})\|^2 \leq IM_1\gamma \quad \text{and} \quad \liminf_{n \to +\infty} \sum_{i \in \mathscr{I}} f_i(y_{n,i}) \leq f^\star,$$

where $M_1 := \sup\{(2/I)\sum_{i \in \mathscr{I}} (f_i(x) - f_i(y_{n,i})) : n \in \mathbb{N}\} < +\infty$ for some $x \in X$ and $f^\star$ is the optimal value of Problem 1.1.

Proof. Let $x \in X$ be fixed arbitrarily. The definition of $\partial f_i(x)$ and the Cauchy-Schwarz inequality imply that, for all $i \in \mathscr{I}$, for all $n \in \mathbb{N}$, and for all $u_i \in \partial f_i(x)$,

$$f_i(x) - f_i(y_{n,i}) \leq \langle x - y_{n,i}, u_i \rangle \leq \|y_{n,i} - x\| \, \|u_i\|,$$

which, together with $\tilde{B} := \max_{i \in \mathscr{I}} \sup\{\|y_{n,i} - x\| \colon n \in \mathbb{N}\} < +\infty$ (by Assumption 3.1), implies that

$$M_1 := \sup\left\{ \frac{2}{I} \sum_{i \in \mathscr{I}} (f_i(x) - f_i(y_{n,i})) \colon n \in \mathbb{N} \right\} \leq 2\tilde{B} \max_{i \in \mathscr{I}} \|u_i\| < +\infty. \tag{3.5}$$

We first show that

$$\liminf_{n \to +\infty} \sum_{i \in \mathscr{I}} \underbrace{\left\{ \|x_n - y_{n,i}\|^2 + \|x_{n,i} - y_{n,i}\|^2 \right\}}_{X_{n,i}} \leq IM_1\gamma. \tag{3.6}$$

If (3.6) does not hold, there exists $\delta > 0$ such that

$$\liminf_{n \to +\infty} \sum_{i \in \mathscr{I}} X_{n,i} > IM_1\gamma + 2\delta.$$

Accordingly, the property of the limit inferior of $(\sum_{i \in \mathscr{I}} \{\|x_n - y_{n,i}\|^2 + \|x_{n,i} - y_{n,i}\|^2\})_{n \in \mathbb{N}}$ ensures that $n_0 \in \mathbb{N}$ exists such that, for all $n \geq n_0$,

$$\sum_{i \in \mathscr{I}} X_{n,i} > IM_1\gamma + \delta. \tag{3.7}$$

Accordingly, Lemma 3.1 with $\gamma_n := \gamma$ $(n \in \mathbb{N})$ guarantees that, for all $n \geq n_0$,

$$\begin{aligned}
\|x_{n+1} - x\|^2 &\leq \|x_n - x\|^2 - \frac{1}{I} \sum_{i \in \mathscr{I}} X_{n,i} + \frac{2}{I}\gamma \sum_{i \in \mathscr{I}} (f_i(x) - f_i(y_{n,i})) \\
&< \|x_n - x\|^2 - \frac{1}{I}(IM_1\gamma + \delta) + M_1\gamma \\
&= \|x_n - x\|^2 - \frac{\delta}{I} \\
&< \|x_{n_0} - x\|^2 - \frac{\delta}{I}(n + 1 - n_0).
\end{aligned}$$

The right side of the above inequality approaches minus infinity as n diverges. Hence, we have a contradiction. This implies that (3.6) holds. Therefore,

$$\liminf_{n \to +\infty} \sum_{i \in \mathscr{I}} \|y_{n,i} - x_{n,i}\|^2 = \liminf_{n \to +\infty} \sum_{i \in \mathscr{I}} \|y_{n,i} - Q_i(y_{n,i})\|^2 \leq IM_1\gamma.$$

Next, we show that

$$\liminf_{n \to +\infty} \sum_{i \in \mathscr{I}} f_i(y_{n,i}) \leq f^\star. \tag{3.8}$$

Assume that (3.8) does not hold. An argument similar to the one for obtaining (3.7) implies that there exist $\zeta > 0$ and $m_0 \in \mathbb{N}$ such that, for all $n \geq m_0$,

$$\sum_{i \in \mathscr{I}} f_i(y_{n,i}) - f^\star > \zeta.$$

Lemma 3.1 thus ensures that, for all $n \geq m_0$ and for all $x^\star \in X^\star := \{x^\star \in X \colon f(x^\star) = f^\star = \inf_{x \in X} f(x)\}$,

$$\begin{aligned}\|x_{n+1} - x^\star\|^2 &\leq \|x_n - x^\star\|^2 + \frac{2}{I}\gamma\left(f^\star - \sum_{i \in \mathscr{I}} f_i(y_{n,i})\right)\\ &< \|x_n - x^\star\|^2 - \frac{2}{I}\gamma\zeta\\ &< \|x_{m_0} - x^\star\|^2 - \frac{2}{I}\gamma\zeta(n+1-m_0),\end{aligned}$$

which is a contradiction. Accordingly, (3.8) holds. This completes the proof. □

3.2. **Diminishing step-size rule.** The following is a convergence analysis of Algorithm 1 with a diminishing step size.

Theorem 3.2. *Suppose that (A1), (A2), and Assumption 3.1 hold and* $\mathrm{Id} - Q_i$ $(i \in \mathscr{I})$ *is demiclosed.** *Let* $(x_n)_{n \in \mathbb{N}}$ *be the sequence generated by Algorithm 1 with* $(\gamma_n)_{n \in \mathbb{N}}$ *satisfying* $\lim_{n \to +\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{+\infty} \gamma_n = +\infty$. *Then, there exists a subsequence of each of* $(x_n)_{n \in \mathbb{N}}$, $(x_{n,i})_{n \in \mathbb{N}}$, *and* $(y_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ *that weakly converges to a solution of Problem 1.1. Moreover,* $(x_n)_{n \in \mathbb{N}}$, $(x_{n,i})_{n \in \mathbb{N}}$, *and* $(y_{n,i})_{n \in \mathbb{N}}$ $(i \in \mathscr{I})$ *strongly converge to a unique solution of Problem 1.1 if one of the following holds:*

(i) *One* f_i *is strongly convex;*
(ii) H *is finite-dimensional, and one* f_i *is strictly convex.*

Proof. We consider two cases.

Case 1: Suppose that there exists $m_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$ and for all $x^\star \in X^\star$, $n \geq m_0$ implies $\|x_{n+1} - x^\star\| \leq \|x_n - x^\star\|$, where $X^\star := \{x^\star \in X \colon f(x^\star) = f^\star = \inf_{x \in X} f(x)\}$. Then, there exists $c := \lim_{n \to +\infty} \|x_n - x^\star\|$. Let $x^\star \in X^\star$ be fixed arbitrarily. Lemma 3.1, together with a discussion similar to that of (3.5), guarantees that there exists

$$M_2 := \sup\left\{\frac{2}{I}\sum_{i \in \mathscr{I}}(f_i(x^\star) - f_i(y_{n,i})) \colon n \in \mathbb{N}\right\} < +\infty$$

such that, for all $n \geq m_0$,

$$\frac{1}{I}\sum_{i \in \mathscr{I}}\left\{\|x_n - y_{n,i}\|^2 + \|x_{n,i} - y_{n,i}\|^2\right\} \leq \|x_n - x^\star\|^2 - \|x_{n+1} - x^\star\|^2 + M_2\gamma_n. \tag{3.9}$$

Accordingly, the conditions $\lim_{n \to +\infty} \gamma_n = 0$ and $c := \lim_{n \to +\infty} \|x_n - x^\star\|$ mean that

$$\lim_{n \to +\infty} \|x_n - y_{n,i}\| = 0 \text{ and } \lim_{n \to +\infty} \|x_{n,i} - y_{n,i}\| = 0 \ (i \in \mathscr{I}). \tag{3.10}$$

From Lemma 3.1, for all $x \in X$ and for all $k \in \mathbb{N}$,

$$\frac{2}{I}\gamma_k \underbrace{\sum_{i \in \mathscr{I}}(f_i(y_{k,i}) - f_i(x))}_{N_k(x)} \leq \|x_k - x\|^2 - \|x_{k+1} - x\|^2, \tag{3.11}$$

which implies that, for all $n \in \mathbb{N}$ and for all $x \in X$,

$$\frac{2}{I}\sum_{k=0}^{n}\gamma_k N_k(x) \leq \|x_0 - x\|^2 - \|x_{n+1} - x\|^2 \leq \|x_0 - x\|^2.$$

* See Section 4 for an example in which Q_i is quasi-firmly nonexpansive and $\mathrm{Id} - Q_i$ is demiclosed.

Accordingly, for all $x \in X$,

$$\frac{2}{I}\sum_{k=0}^{+\infty} \gamma_k N_k(x) < +\infty. \tag{3.12}$$

Here, we show that, for all $x \in X$,

$$\liminf_{n\to+\infty} N_n(x) \leq 0. \tag{3.13}$$

Assume that (3.13) does not hold; i.e., there exists $x_0 \in X$ such that $\liminf_{n\to+\infty} N_n(x_0) > 0$. Then, $m_1 \in \mathbb{N}$ and $\theta > 0$ exist such that, for all $n \geq m_1$, $N_n(x_0) \geq \theta$. From (3.12) and $\sum_{n=0}^{+\infty} \gamma_n = +\infty$,

$$+\infty = \frac{2\theta}{I}\sum_{k=m_1}^{+\infty} \gamma_k \leq \frac{2}{I}\sum_{k=m_1}^{+\infty} \gamma_k N_k(x_0) < +\infty,$$

which is a contradiction. Hence, (3.13) holds, i.e., for all $x \in X$,

$$\liminf_{n\to+\infty} \sum_{i\in\mathscr{I}} f_i(y_{n,i}) \leq \sum_{i\in\mathscr{I}} f_i(x) =: f(x). \tag{3.14}$$

The definition of $u_{n,i} \in \partial f_i(x_n)$ and the Cauchy-Schwarz inequality ensure that, for all $i \in \mathscr{I}$ and for all $n \in \mathbb{N}$,

$$f_i(x_n) - f_i(y_{n,i}) \leq \langle x_n - y_{n,i}, u_{n,i}\rangle \leq \|x_n - y_{n,i}\| \, \|u_{n,i}\|.$$

Proposition 2.1(iii) and the boundedness of $(x_n)_{n\in\mathbb{N}}$ (see also Lemma 3.2) guarantee that there exists $B_1 := \max_{i\in\mathscr{I}} \sup\{\|u_{n,i}\| : n \in \mathbb{N}\} < +\infty$ such that, for all $n \in \mathbb{N}$,

$$f(x_n) = \sum_{i\in\mathscr{I}} f_i(x_n) \leq B_1 \sum_{i\in\mathscr{I}} \|x_n - y_{n,i}\| + \sum_{i\in\mathscr{I}} f_i(y_{n,i}).$$

Therefore, (3.10) and (3.14) lead to the finding that, for all $x \in X$,

$$\liminf_{n\to+\infty} f(x_n) \leq B_1 \lim_{n\to+\infty} \sum_{i\in\mathscr{I}} \|x_n - y_{n,i}\| + \liminf_{n\to+\infty} \sum_{i\in\mathscr{I}} f_i(y_{n,i}) \leq f(x). \tag{3.15}$$

Accordingly, a subsequence $(x_{n_l})_{l\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ exists such that, for all $x \in X$,

$$\lim_{l\to\infty} f\left(x_{n_l}\right) = \liminf_{n\to\infty} f\left(x_n\right) \leq f(x). \tag{3.16}$$

Since $(x_{n_l})_{l\in\mathbb{N}}$ is bounded (see also Lemma 3.2), there exists $(x_{n_{l_m}})_{m\in\mathbb{N}}$ ($\subset (x_{n_l})_{l\in\mathbb{N}}$) such that $(x_{n_{l_m}})_{m\in\mathbb{N}}$ weakly converges to $x_* \in H$. From (3.10), $(y_{n_{l_m},i})$ $(i \in \mathscr{I})$ weakly converges to x_*. Hence, (3.10) and the demiclosedness of $\mathrm{Id} - Q_i$ ensure that $x_* \in \mathrm{Fix}(Q_i)$ $(i \in \mathscr{I})$, i.e., $x_* \in X$. Proposition 2.3 ensures that the continuity and convexity of f (by (A2)) imply that f is weakly lower semicontinuous, which means that $f(x_*) \leq \liminf_{m\to\infty} f(x_{n_{l_m}})$. Therefore, (3.16) leads to the finding that, for all $x \in X$,

$$f\left(x_*\right) \leq \liminf_{m\to\infty} f\left(x_{n_{l_m}}\right) = \lim_{m\to\infty} f\left(x_{n_{l_m}}\right) \leq f(x),$$

that is, $x_* \in X^\star$. Let us take another subsequence $(x_{n_{l_k}})_{k\in\mathbb{N}}$ ($\subset (x_{n_l})_{l\in\mathbb{N}}$) such that $(x_{n_{l_k}})_{k\in\mathbb{N}}$ weakly converges to $x_{**} \in H$. A discussion similar to the one for obtaining $x_* \in X^\star$ guarantees that $x_{**} \in X^\star$.

Here, it is proven that $x_* = x_{**}$. Now, let us assume that $x_* \neq x_{**}$. Then, the existence of $c := \lim_{n\to\infty} \|x_n - x^\star\|$ $(x^\star \in X^\star)$ and Proposition 2.2 imply that

$$\begin{aligned} c &= \lim_{m\to\infty} \left\|x_{n_{l_m}} - x_*\right\| < \lim_{m\to\infty} \left\|x_{n_{l_m}} - x_{**}\right\| \\ &= \lim_{n\to\infty} \|x_n - x_{**}\| = \lim_{k\to\infty} \left\|x_{n_{l_k}} - x_{**}\right\| < \lim_{k\to\infty} \left\|x_{n_{l_k}} - x_*\right\| \\ &= c, \end{aligned}$$

which is a contradiction. Hence, $x_* = x_{**}$. Accordingly, any subsequence of $(x_{n_l})_{l\in\mathbb{N}}$ converges weakly to $x_* \in X^\star$; i.e., $(x_{n_l})_{l\in\mathbb{N}}$ converges weakly to $x_* \in X^\star$. This means that x_* is a weak cluster point of $(x_n)_{n\in\mathbb{N}}$ and belongs to $X^\star$. A discussion similar to the one for obtaining $x_* = x_{**}$ guarantees that there is only one weak cluster point of $(x_n)_{n\in\mathbb{N}}$, so we can conclude that, in Case 1, $(x_n)_{n\in\mathbb{N}}$ weakly converges to a point in $X^\star$.

Case 2: Suppose that, for all $m \in \mathbb{N}$, there exist $n \in \mathbb{N}$ and $x_0^\star \in X^\star$ such that $n \geq m$ and $\|x_{n+1} - x_0^\star\| > \|x_n - x_0^\star\|$. This implies that $(x_{n_j})_{j\in\mathbb{N}}$ $(\subset (x_n)_{n\in\mathbb{N}})$ exists such that, for all $j \in \mathbb{N}$, $\|x_{n_j+1} - x_0^\star\| > \|x_{n_j} - x_0^\star\| =: \Gamma_{n_j}$. Proposition 2.4 thus guarantees that $m_1 \in \mathbb{N}$ exists such that, for all $n \geq m_1$, $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, where $\tau(n)$ is defined as in Proposition 2.4. From Lemma 3.1 (see also (3.9)), for all $n \geq m_1$,

$$\frac{1}{I}\sum_{i\in\mathscr{I}} \left\{ \left\|x_{\tau(n)} - y_{\tau(n),i}\right\|^2 + \left\|x_{\tau(n),i} - y_{\tau(n),i}\right\|^2 \right\} \leq \Gamma_{\tau(n)}^2 - \Gamma_{\tau(n)+1}^2 + \tilde{M}_2\gamma_{\tau(n)} \leq \tilde{M}_2\gamma_{\tau(n)},$$

where

$$\tilde{M}_2 := \sup\left\{ \frac{2}{I}\sum_{i\in\mathscr{I}} \left(f_i(x^\star) - f_i(y_{\tau(n),i})\right) : n \in \mathbb{N} \right\}$$

is finite by Assumption 3.1 (see also (3.5)). Hence, the condition $\lim_{n\to+\infty} \gamma_{\tau(n)} = 0$ implies that

$$\lim_{n\to+\infty} \left\|x_{\tau(n)} - y_{\tau(n),i}\right\| = 0 \text{ and } \lim_{n\to+\infty} \left\|x_{\tau(n),i} - y_{\tau(n),i}\right\| = 0 \ (i \in \mathscr{I}). \tag{3.17}$$

From (3.11), for all $n \geq m_1$,

$$\frac{2}{I}\gamma_{\tau(n)} N_{\tau(n)}(x_0^\star) \leq \Gamma_{\tau(n)}^2 - \Gamma_{\tau(n)+1}^2 \leq 0,$$

which, together with $\gamma_{\tau(n)} \geq 0$ $(n \geq m_1)$, implies that $N_{\tau(n)}(x_0^\star) \leq 0$. Accordingly,

$$\limsup_{n\to+\infty} \sum_{i\in\mathscr{I}} f_i\left(y_{\tau(n),i}\right) \leq f^\star.$$

An argument similar to the one for obtaining (3.15), together with (3.17), implies that

$$\limsup_{n\to+\infty} f\left(x_{\tau(n)}\right) \leq f^\star.$$

Choose a subsequence $(x_{\tau(n_k)})_{k\in\mathbb{N}}$ of $(x_{\tau(n)})_{n\geq m_1}$ arbitrarily. Then,

$$\limsup_{k\to+\infty} f\left(x_{\tau(n_k)}\right) \leq \limsup_{n\to+\infty} f\left(x_{\tau(n)}\right) \leq f^\star. \tag{3.18}$$

The boundedness of $(x_{\tau(n_k)})_{k\in\mathbb{N}}$ ensures that $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ $(\subset (x_{\tau(n_k)})_{k\in\mathbb{N}})$ exists such that $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ weakly converges to $x_\star \in H$. Then, (3.17) and the demiclosedness of $\mathrm{Id} - Q_i$ ensure that $x_\star \in X$. Moreover, Proposition 2.3 and (3.18) guarantee that

$$f(x_\star) \leq \liminf_{l\to+\infty} f\left(x_{\tau\left(n_{k_l}\right)}\right) \leq \limsup_{l\to+\infty} f\left(x_{\tau\left(n_{k_l}\right)}\right) \leq f^\star,$$

that is, $x_\star \in X^\star$. Therefore, $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ weakly converges to $x_\star \in X^\star$. From Cases 1 and 2, there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ that weakly converges to a point in $X^\star$.

Suppose that assumption (i) in Theorem 3.2 holds. The strong convexity of $f := \sum_{i\in\mathscr{I}} f^{(i)}$ implies that $X^\star$ consists of one point, denoted by $x^\star$. In Case 1, the strong convexity of f guarantees that there exists $\beta > 0$ such that, for all $\alpha \in (0,1)$ and for all $l \in \mathbb{N}$, $(\beta/2)\alpha(1-\alpha)\|x_{n_l} - x^\star\|^2 \leq \alpha f(x_{n_l}) + (1-\alpha)f^\star - f(\alpha x_{n_l} + (1-\alpha)x^\star)$. Accordingly, from the existence of $c := \lim_{n\to+\infty}\|x_n - x^\star\|$ and (3.16), we have

$$\begin{aligned}\frac{\beta}{2}\alpha(1-\alpha)\lim_{l\to+\infty}\|x_{n_l} - x^\star\|^2 &\leq \lim_{l\to+\infty}\left(\alpha f(x_{n_l}) + (1-\alpha)f^\star\right)\\ &\quad + \limsup_{l\to+\infty}\left(-f(\alpha x_{n_l} + (1-\alpha)x^\star)\right)\\ &\leq f^\star - \liminf_{l\to+\infty} f(\alpha x_{n_l} + (1-\alpha)x^\star),\end{aligned}$$

which, together with the weak convergence of $(x_{n_l})_{l\in\mathbb{N}}$ to $x^\star$ and Proposition 2.3, implies that

$$\frac{\beta}{2}\alpha(1-\alpha)\lim_{l\to+\infty}\|x_{n_l} - x^\star\|^2 \leq f^\star - f(\alpha x^\star + (1-\alpha)x^\star) = 0.$$

Hence, $(x_{n_l})_{l\in\mathbb{N}}$ strongly converges to $x^\star$. Therefore, from [2, Theorem 5.11], the whole sequence $(x_n)_{n\in\mathbb{N}}$ strongly converges to $x^\star$. From (3.10), $(x_{n,i})_{n\in\mathbb{N}}$ and $(y_{n,i})_{n\in\mathbb{N}}$ $(i \in \mathscr{I})$ strongly converge to $x^\star$. In Case 2, the strong convexity of f leads to the deduction that, for all $\alpha \in (0,1)$ and for all $l \in \mathbb{N}$,

$$\begin{aligned}\frac{\beta}{2}\alpha(1-\alpha)\limsup_{l\to+\infty}\left\|x_{\tau(n_{k_l})} - x^\star\right\|^2 &\leq \alpha\limsup_{l\to+\infty} f\left(x_{\tau(n_{k_l})}\right) + (1-\alpha)f^\star\\ &\quad - \liminf_{l\to+\infty} f\left(\alpha x_{\tau(n_{k_l})} + (1-\alpha)x^\star\right).\end{aligned}$$

The weak convergence of $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ to $x^\star$, the weakly lower semicontinuity of f (by Proposition 2.3), and (3.18) imply that

$$\frac{\beta}{2}\alpha(1-\alpha)\limsup_{l\to+\infty}\left\|x_{\tau(n_{k_l})} - x^\star\right\|^2 \leq f^\star - f(\alpha x^\star + (1-\alpha)x^\star) = 0,$$

which implies that $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ strongly converges to $x^\star$. When another subsequence $(x_{\tau(n_{k_m})})_{m\in\mathbb{N}}$ $(\subset (x_{\tau(n_k)})_{k\in\mathbb{N}})$ can be chosen, a discussion similar to the one for showing the weak convergence of $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ to a point in $X^\star$ guarantees that $(x_{\tau(n_{k_m})})_{m\in\mathbb{N}}$ also weakly converges to a point in $X^\star$. Furthermore, a discussion similar to the one for showing the strong convergence of $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ to $x^\star$ ensures that $(x_{\tau(n_{k_m})})_{m\in\mathbb{N}}$ strongly converges to the same $x^\star$. Hence, it is guaranteed that $(x_{\tau(n_k)})_{k\in\mathbb{N}}$ strongly converges to $x^\star$. Since $(x_{\tau(n_k)})_{k\in\mathbb{N}}$ is an arbitrary subsequence of $(x_{\tau(n)})_{n\geq m_1}$, $(x_{\tau(n)})_{n\geq m_1}$ strongly converges to $x^\star$; i.e., $\lim_{n\to\infty}\Gamma_{\tau(n)} = \lim_{n\to\infty}\|x_{\tau(n)} - x^\star\| = 0$. Accordingly, Proposition 2.4 ensures that

$$\limsup_{n\to\infty}\|x_n - x^\star\| \leq \limsup_{n\to\infty}\Gamma_{\tau(n)+1} = 0,$$

which implies that, in Case 2, the whole sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x^\star$. Moreover, Lemma 3.1 and $\lim_{n\to+\infty}\gamma_n = 0$ imply that $\lim_{n\to+\infty}\|x_n - y_{n,i}\| = \lim_{n\to+\infty}\|x_{n,i} - y_{n,i}\| = 0$ $(i \in \mathscr{I})$. Therefore, $(x_{n,i})_{n\in\mathbb{N}}$ and $(y_{n,i})_{n\in\mathbb{N}}$ $(i \in \mathscr{I})$ converge to $x^\star$.

Suppose that assumption (ii) in Theorem 3.2 holds. Let $x^\star \in X^\star$ be the unique solution to Problem 1.1. In Case 1, it is guaranteed that $(x_n)_{n\in\mathbb{N}}$ converges to $x^\star \in X^\star$. From (3.10), $(x_{n,i})_{n\in\mathbb{N}}$ and $(y_{n,i})_{n\in\mathbb{N}}$ $(i \in \mathscr{I})$ strongly converge to $x^\star$. Moreover, in Case 2, the convergence of $(x_{\tau(n_{k_l})})_{l\in\mathbb{N}}$ to $x^\star$ is guaranteed. A discussion similar to the one for showing the strong convergence of $(x_{\tau(n)})_{n\geq m_1}$ to $x^\star$ ensures that

$(x_{\tau(n)})_{n \geq m_1}$ converges to $x^\star \in X^\star$. Proposition 2.4 thus guarantees that the whole sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x^\star$. Lemma 3.1 and $\lim_{n\to+\infty}\gamma_n = 0$ imply that $\lim_{n\to+\infty}\|x_n - y_{n,i}\| = \lim_{n\to+\infty}\|x_{n,i} - y_{n,i}\| = 0$ $(i \in \mathscr{I})$. Therefore, $(x_{n,i})_{n\in\mathbb{N}}$ and $(y_{n,i})_{n\in\mathbb{N}}$ $(i \in \mathscr{I})$ converge to $x^\star$. This completes the proof. □

The following is a convergence-rate analysis of Algorithm 1 with a diminishing step size.

Theorem 3.3. *Suppose that the assumptions in Theorem 3.1 hold and a monotone decreasing sequence* $(\gamma_n)_{n\in\mathbb{N}}$ *satisfies* $\lim_{n\to+\infty}\gamma_n = 0$, $\lim_{n\to+\infty}(n\gamma_n)^{-1} = 0$, $\sum_{n=0}^{+\infty}\gamma_n = +\infty$, *and* $\lim_{n\to+\infty} n^{-1}\sum_{k=0}^{n-1}\gamma_k = 0$. *Then, Algorithm 1 satisfies that, for all* $n \geq 1$,

$$\sum_{i\in\mathscr{I}}\left(\frac{1}{n}\sum_{k=0}^{n-1}\|y_{k,i} - Q_i(y_{k,i})\|^2\right) \leq \frac{I\|x_0 - x\|^2}{n} + \frac{\tilde{M}_1}{n}\sum_{k=0}^{n-1}\gamma_k \quad \text{and} \quad \sum_{i\in\mathscr{I}} f_i\left(\frac{1}{n}\sum_{k=0}^{n-1} y_{k,i}\right) \leq f^\star + \frac{IB}{2n\gamma_n},$$

where $x^\star$ *is a solution of Problem 1.1,* $\tilde{M}_1 := \sup\{2\sum_{i\in\mathscr{I}}(f_i(x^\star) - f_i(y_{n,i}))\colon n \in \mathbb{N}\} < +\infty$, *and* $B :=$ $\sup\{\|x_n - x^\star\|^2\colon n \in \mathbb{N}\} < +\infty$.

Proof. Let $x^\star \in X^\star$. Lemma 3.1 implies that, for all $n \geq 1$,

$$\frac{1}{I}\sum_{i\in\mathscr{I}}\sum_{k=0}^{n-1}\left\{\|x_k - y_{k,i}\|^2 + \|x_{k,i} - y_{k,i}\|^2\right\} \leq \|x_0 - x\|^2 + \frac{\tilde{M}_1}{I}\sum_{k=0}^{n-1}\gamma_k,$$

which in turn implies that

$$\sum_{i\in\mathscr{I}}\left(\frac{1}{n}\sum_{k=0}^{n-1}\|x_{k,i} - y_{k,i}\|^2\right) \leq \frac{1}{n}\sum_{i\in\mathscr{I}}\sum_{k=0}^{n-1}\left\{\|x_k - y_{k,i}\|^2 + \|x_{k,i} - y_{k,i}\|^2\right\} \leq \frac{I\|x_0 - x\|^2}{n} + \frac{\tilde{M}_1}{n}\sum_{k=0}^{n-1}\gamma_k.$$

Lemma 3.1 indicates that, for all $k \in \mathbb{N}$,

$$\sum_{i\in\mathscr{I}} f_i(y_{k,i}) - f^\star \leq \frac{I}{2\gamma_k}\left\{\|x_k - x^\star\|^2 - \|x_{k+1} - x^\star\|^2\right\}.$$

Summing the above inequality from $k = 0$ to $k = n-1$ implies that, for all $n \geq 1$,

$$\frac{1}{n}\sum_{k=0}^{n-1}\sum_{i\in\mathscr{I}} f_i(y_{k,i}) - f^\star \leq \frac{I}{2n}\underbrace{\sum_{k=0}^{n-1}\frac{1}{\gamma_k}\left\{\|x_k - x^\star\|^2 - \|x_{k+1} - x^\star\|^2\right\}}_{X_n}.$$

The definition of X_n means that

$$X_n = \frac{\|x_0 - x^\star\|}{\gamma_0} + \sum_{k=1}^{n-1}\left\{\frac{\|x_k - x^\star\|^2}{\gamma_k} - \frac{\|x_k - x^\star\|^2}{\gamma_{k-1}}\right\} - \frac{\|x_n - x^\star\|^2}{\gamma_{n-1}},$$

which, together with $\gamma_n \leq \gamma_{n-1}$ $(n \geq 1)$ and $B := \sup\{\|x_n - x^\star\|^2\colon n \in \mathbb{N}\} < +\infty$ (by Lemma 3.2), implies that

$$X_n \leq \frac{B}{\gamma_0} + B\sum_{k=1}^{n-1}\left(\frac{1}{\gamma_k} - \frac{1}{\gamma_{k-1}}\right) = \frac{B}{\gamma_{n-1}} \leq \frac{B}{\gamma_n}.$$

The convexity of f_i thus ensures that, for all $n \geq 1$,

$$\sum_{i\in\mathscr{I}} f_i\left(\frac{1}{n}\sum_{k=0}^{n-1} y_{k,i}\right) - f^\star \leq \frac{IB}{2n\gamma_n},$$

which completes the proof. □

Let us consider the rate of convergence of Algorithm 1 with $\gamma_n := n^{-1/2}$ $(n \geq 1)$. The step size $(\gamma_n)_{n\geq 1}$ is monotone decreasing and satisfies $\lim_{n\to+\infty}\gamma_n = 0$, $\lim_{n\to+\infty}(n\gamma_n)^{-1} = 0$, and $\sum_{n=0}^{+\infty}\gamma_n = +\infty$. Moreover, the Cauchy-Schwarz inequality and $\sum_{k=0}^{n-1} k^{-1} \leq 1+\ln n$ mean that

$$\frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{\sqrt{k}} \leq \frac{\sqrt{n}}{n}\sqrt{\sum_{k=0}^{n-1}\frac{1}{k}} \leq \sqrt{\frac{1+\ln n}{n}},$$

which implies that $\lim_{n\to+\infty} n^{-1}\sum_{k=0}^{n-1}\gamma_k = 0$. Theorem 3.3 indicates that Algorithm 1 with $\gamma_n := n^{-1/2}$ satisfies that, for all $n \geq 1$,

$$\sum_{i\in\mathscr{I}}\left(\frac{1}{n}\sum_{k=0}^{n-1}\|y_{k,i} - Q_i(y_{k,i})\|^2\right) = \mathscr{O}\left(\sqrt{\frac{1+\ln n}{n}}\right) \quad \text{and} \quad \sum_{i\in\mathscr{I}} f_i\left(\frac{1}{n}\sum_{k=0}^{n-1} y_{k,i}\right) \leq f^\star + \frac{IB}{2\sqrt{n}},$$

where $\mathscr{O}$ stands for the Landau notation (see [10] for a convergence rate analysis of stochastic approximation methods).

4. Numerical Comparisons

Let us compare the performance of Algorithm 1 with the one of the existing parallel subgradient method (PSM) [7, Algorithm 3.1] (see (3.1)) and incremental subgradient method (ISM) [7, Algorithm 4.1] for the following problem (see also [7, Problem 5.1]): Let $a_{i,j} > 0$, $b_{i,j}, d_i \in \mathbb{R}$ $(i \in \mathscr{I}, j = 1,2,\ldots,N)$, and $c_i := (c_{i,j})_{j=1}^N \in \mathbb{R}^N$ $(i \in \mathscr{I})$ with $c_{i,j} > 0$. Then,

$$\text{minimize } f(x) := \sum_{i\in\mathscr{I}} f_i(x) \text{ subject to } x \in X := \bigcap_{i\in\mathscr{I}} \mathrm{Fix}(Q_i) = \bigcap_{i\in\mathscr{I}} \mathrm{lev}_{\leq 0} g_i, \tag{4.1}$$

where $f_i \colon \mathbb{R}^N \to \mathbb{R}$ and $Q_i \colon \mathbb{R}^N \to \mathbb{R}^N$ are defined for all $x := (x_j)_{j=1}^N \in \mathbb{R}^N$ by

$$f_i(x) := \sum_{j=1}^N a_{i,j}\left|x_j - b_{i,j}\right| \text{ and } Q_i(x) := \begin{cases} x - \dfrac{g_i(x)}{\|z_i(x)\|^2} z_i(x) & \text{if } g_i(x) > 0, \\ x & \text{if } x \in \mathrm{lev}_{\leq 0} g_i := \{x \in \mathbb{R}^N \colon g_i(x) \leq 0\}, \end{cases}$$

$g_i \colon \mathbb{R}^N \to \mathbb{R}$ is defined for all $x \in \mathbb{R}^N$ by

$$g_i(x) := \begin{cases} \langle c_i, x\rangle + d_i & \text{if } \langle c_i, x\rangle > -d_i, \\ 0 & \text{otherwise,} \end{cases}$$

and $z_i(x)$ is any vector in $\partial g_i(x)$. The above mapping Q_i is called the *subgradient projection* related to g_i. Q_i satisfies quasi-firm nonexpnasivity, and $\mathrm{Id} - Q_i$ satisfies the demiclosedness condition [1, Lemma 3.1].

The experiment was conducted on a MacBook Air (13-inch, 2017) with a 1.8 GHz Intel (R) Core (TM) i5 CPU processor, 8 GB, 1600 MHz DDR3 memory, and Mac OS Catalina (Version 10.15) operating system. PSM, ISM, and Algorithm 1 were written in Python 3.7.4 with the NumPy 1.17.2 package. We set $I = 256$ and $N = 1000$ and randomly chose $a_{i,j} \in (0,100]$, $b_{i,j} \in [-100,100)$, $d_i \in [-1,0)$, and $c_{i,j} \in [-0.5,0.5)$. The stopping condition was $n = 10000$. The step sizes were as follows:

$$\text{Constant step sizes: } \gamma_n := 10^{-1}, 10^{-3},$$

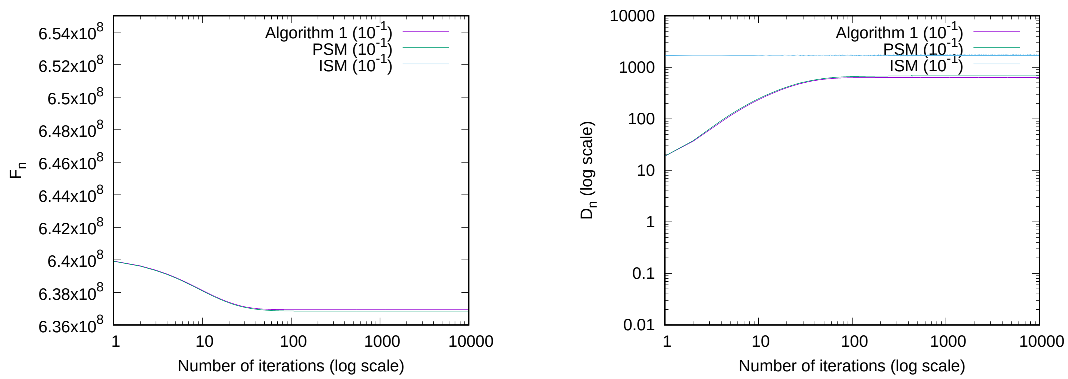
$$\text{Diminishing step sizes: } \gamma_n := \frac{10^{-1}}{n+1}, \frac{10^{-3}}{n+1}.$$

The performance measures were as follows: for $n \in \mathbb{N}$,

$$F_n := \frac{1}{10} \sum_{s=1}^{10} \sum_{i \in \mathscr{I}} f_i(x_n(s)) \text{ and } D_n := \frac{1}{10} \sum_{s=1}^{10} \sum_{i \in \mathscr{I}} \| x_n(s) - Q_i(x_n(s)) \| ,$$

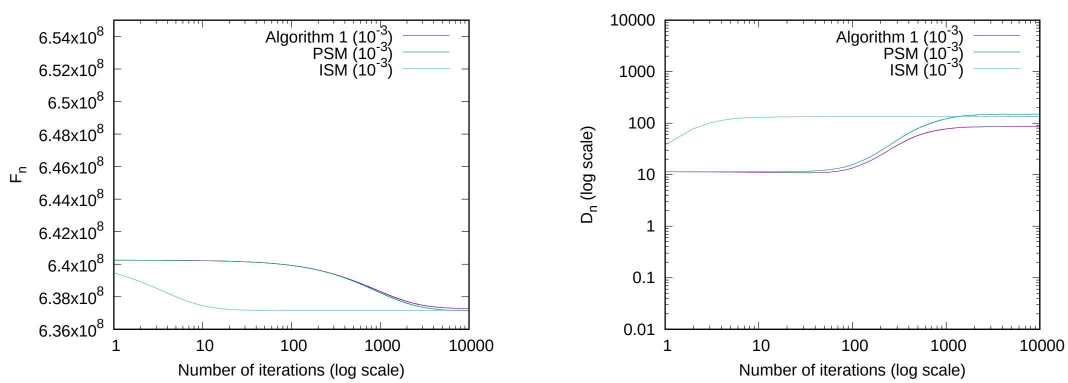
where $(x_n(s))_{n \in \mathbb{N}}$ is the sequence generated by each of the three algorithms with the randomly chosen initial point $x_0(s) \in [0,1)^N$ $(s = 1, 2, \ldots, 10)$. If $(D_n)_{n \in \mathbb{N}}$ converges to 0, the algorithms converge to a point in X.

Figure 1 shows that the algorithms with $\gamma_n = \lambda_n = 10^{-1}$ did not converge to a point in X. Figure 2 indicates that, although the values of D_{10000} generated by the algorithms with $\gamma_n = \lambda_n = 10^{-3}$ were less than those generated by the algorithms with $\gamma_n = \lambda_n = 10^{-1}$, the algorithms with $\gamma_n = \lambda_n = 10^{-3}$ did not converge to a point in X. These results imply that it would be difficult to set an appropriate constant step size in advance.



(A) Evaluation of F_n (B) Evaluation of D_n

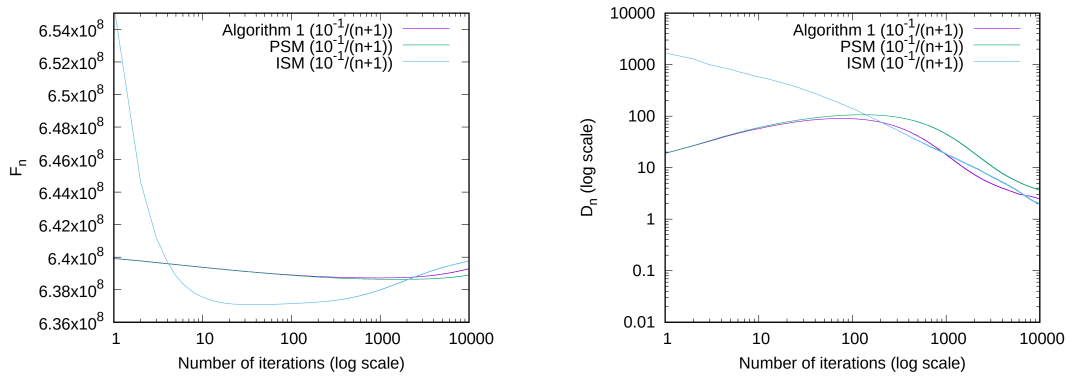
FIGURE 1. Behaviors of F_n and D_n for Algorithm 1, PSM, and ISM with $\gamma_n = \lambda_n = 10^{-1}$



(A) Evaluation of F_n (B) Evaluation of D_n

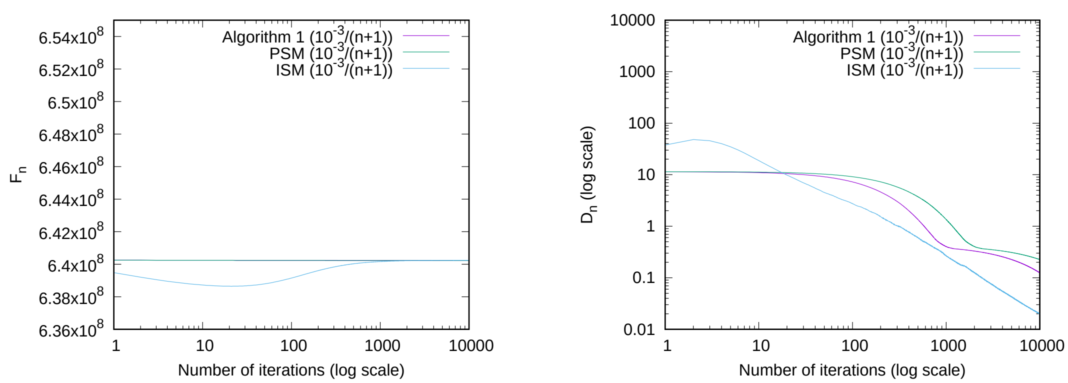
FIGURE 2. Behaviors of F_n and D_n for Algorithm 1, PSM, and ISM with $\gamma_n = \lambda_n = 10^{-3}$

Meanwhile, Figures 3 and 4 show that Algorithm 1 with diminishing step sizes $\gamma_n = 10^{-1}/(n+1), 10^{-3}/(n+1)$ converged to a point in X, as guaranteed by Theorem 3.2. These figures also show that F_n remains stable. Accordingly, from Theorem 3.2, Algorithm 1 converged to a solution of problem (4.1). Figures 3 and 4 also indicate that Algorithm 1 performs comparably to PSM and ISM.



(A) Evaluation of F_n (B) Evaluation of D_n

FIGURE 3. Behaviors of F_n and D_n for Algorithm 1, PSM, and ISM with $\gamma_n = \lambda_n = 10^{-1}/(n+1)$



(A) Evaluation of F_n (B) Evaluation of D_n

FIGURE 4. Behaviors of F_n and D_n for Algorithm 1, PSM, and ISM with $\gamma_n = \lambda_n = 10^{-3}/(n+1)$

5. CONCLUSION

This paper presented a parallel proximal method for solving the minimization problem of the sum of convex functions over the intersection of fixed point sets of quasi-nonexpansive mappings in a real Hilbert space. It also provided convergence and convergence-rate analyses. Numerical comparisons showed that the performance of the algorithm is almost the same as those of the existing methods.

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