Optimization for Inconsistent Split Feasibility Problems[0]

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Abstract: The split feasibility problem deals with finding a point in a closed convex subset of the domain space of a linear operator such that the image of the point under the linear operator is in a prescribed closed convex subset of the image space. The split feasibility problem and its variants and generalizations have been widely investigated as a means for resolving practical inverse problems in various disciplines. Many iterative algorithms have been proposed for solving the problem. This paper discusses a split feasibility problem which does not have a solution, referred to as an inconsistent split feasibility problem. When the closed convex set of the domain space is the absolute set and the closed convex set of the image space is the subsidiary set, it would be reasonable to formulate a compromise solution of the inconsistent split feasibility problem by using a point in the absolute set such that its image of the linear operator is closest to the subsidiary set in terms of the norm. We show that the problem of finding the compromise solution can be expressed as a convex minimization problem over the fixed point set of a nonexpansive mapping and propose an iterative algorithm, with three-term conjugate gradient directions, for solving the minimization problem.

Keywords: convex optimization; fixed point; inconsistent split feasibility problem; nonexpansive mapping; three-term conjugate gradient method
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1 Introduction

Inverse problems in various disciplines can be expressed as split feasibility problems and its generalizations, such as the multiple-set split feasibility problem and split common fixed point problem (see, e.g., [4, 5, 6, 7] and references therein), and many iterative algorithms have been presented to solve these problems. First, let us describe the multiple-set split feasibility problem (MSFP).[1]

$$\text{Find} \quad x^\star \in C := \bigcap_{i \in \mathcal{I}} C^{(i)} \quad \text{such that} \quad Ax^\star \in Q := \bigcap_{j \in \mathcal{J}} Q^{(j)}, \tag{1.1}$$

where $C^{(i)}$ $(\subset \mathbb{R}^N)$ $(i \in \mathcal{I} := \{1, 2, \ldots, I\})$ and $Q^{(j)}$ $(\subset \mathbb{R}^M)$ $(j \in \mathcal{J} := \{1, 2, \ldots, J\})$ are nonempty, closed, and convex, and $A \in \mathbb{R}^{M \times N}$. The conventional algorithms (see, e.g., [4, 6, 28, 31, 35, 36] and references therein) for solving MSFP can be applied if it is known from the beginning that MSFP (1.1) has a solution. However, it would be difficult to verify whether MSFP (1.1) has a solution or not before executing the conventional algorithms. This implies the applications of the conventional algorithms are severely limited. Therefore, we should devise algorithms that work without having to assume the existence of a solution to MSFP (1.1). This paper deals with an *inconsistent multiple-set split feasibility problem* (IMSFP) under following assumptions.

(I) $C^{(i)}$ $(i \in \mathcal{I})$ are the absolute sets for which the conditions must be satisfied, whereas $Q^{(j)}$ and $D^{(j)} := \{x \in \mathbb{R}^N : Ax \in Q^{(j)}\}$ $(j \in \mathcal{J})$ are the subsidiary sets for which the conditions are satisfied as much as possible. We assume that $\bigcap_{i \in \mathcal{I}} C^{(i)} \cap \bigcap_{j \in \mathcal{J}} D^{(j)} = \emptyset$.[2]

(II) We can use a nonexpansive mapping, $T^{(i)} : \mathbb{R}^N \to \mathbb{R}^N$, satisfying $\mathrm{Fix}(T^{(i)}) := \{x \in \mathbb{R}^N : T^{(i)}(x) = x\} = C^{(i)}$[3] and the metric projection onto $Q^{(j)}$ $(j \in \mathcal{J})$, denoted by $P_{Q^{(j)}}$.[4]

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[1]MSFP (1.1) when $I = J = 1$ is referred to as the split feasibility problem.

[2]The condition, $\bigcap_{i \in \mathcal{I}} C^{(i)} \cap \bigcap_{j \in \mathcal{J}} D^{(j)} = \emptyset$, implies that MSFP (1.1) has no solution.

[3]$T : \mathbb{R}^N \to \mathbb{R}^N$ is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ $(x, y \in \mathbb{R}^N)$, where $\|\cdot\|$ stands for the Euclidean norm. $\mathrm{Fix}(T)$ is closed and convex when T is nonexpansive [14, Proposition 5.3].

[4]Given a closed convex set C $(\subset \mathbb{R}^N)$, the metric projection onto C is defined as follows: $P_C(x) \in C$

The following is an example of $T^{(i)}$ satisfying assumption (II). When a closed convex set $C_k^{(i)}$ ($k = 1,2,\ldots,m(i)$) is simple in the sense that $P_{C_k^{(i)}}$ can be computed within a finite number of arithmetic operations (e.g., $C_k^{(i)}$ is a closed ball, a closed cone, or a half-space) and $C^{(i)} := \bigcap_{k=1}^{m(i)} C_k^{(i)}$, we can use a nonexpansive mapping $T^{(i)} := \sum_{k=1}^{m(i)} u_k^{(i)} P_{C_k^{(i)}}$, where $(u_k^{(i)})_{k=1}^{m(i)} \subset (0,1)$ satisfies $\sum_{k=1}^{m(i)} u_k^{(i)} = 1$, with $\mathrm{Fix}(T^{(i)}) = \bigcap_{k=1}^{m(i)} C_k^{(i)} = C^{(i)}$ [29, Fact 2.1(b), (c)].

Since $C = \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$ is the absolute set, it would be reasonable to define a compromise solution of IMSFP as a point in $\bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$ such that its image of A is closest to $Q^{(j)}$ ($j \in \mathcal{J}$) in terms of the mean square norm (see [10, section I, Framework 2] and [29, Definition 4.1] for the details of the compromise solution). The mean square value of the distances from Ax ($\in \mathbb{R}^M$) to $Q^{(j)}$ ($j \in \mathcal{J}$) is represented as follows: given $(w^{(j)})_{j\in\mathcal{J}} \subset (0,1)$ satisfying $\sum_{j\in\mathcal{J}} w^{(j)} = 1$,

$$f_D(x) := \frac{1}{2} \sum_{j\in\mathcal{J}} w^{(j)} \left\| P_{Q^{(j)}}(Ax) - Ax \right\|^2 \quad \left(x \in \mathbb{R}^N \right). \tag{1.2}$$

Hence, the compromise solution of IMSFP can be found by taking the minimizer of f_D over $\bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$. Therefore, the main objective of this paper is to solve the following problem.

$$\text{Find} \quad x^\star \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right) \quad \text{such that} \quad f_D(x^\star) = \min_{x\in\bigcap_{i\in\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right)} f_D(x). \tag{1.3}$$

Even if $C \cap D := \bigcap_{i\in\mathcal{I}} C^{(i)} \cap \bigcap_{j\in\mathcal{J}} D^{(j)} = \emptyset$, the solution set of problem (1.3) is well defined because it is the set of all minimizers of f_D over $\bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$. When at least one of the fixed point sets $\mathrm{Fix}(T^{(i)})$ ($i \in \mathcal{I}$) is bounded, the continuity of f_D guarantees that problem (1.3) has a solution. Moreover, if MSFP (1.1) has a solution; i.e., $C \cap D \neq \emptyset$, we find that

$$\begin{aligned} C \cap D &= C \cap \bigcap_{j\in\mathcal{J}} \left\{ x \in \mathbb{R}^N : Ax \in Q^{(j)} \right\} \\ &= C \cap \bigcap_{j\in\mathcal{J}} \left\{ x \in \mathbb{R}^N : \left\| P_{Q^{(j)}}(Ax) - Ax \right\| = 0 \right\} \\ &= \bigcap_{i\in\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right) \cap \left\{ x \in \mathbb{R}^N : f_D(x) = \min_{y\in\mathbb{R}^N} f_D(y) \right\} \\ &= \left\{ x^\star \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right) : f_D(x^\star) = \min_{x\in\bigcap_{i\in\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right)} f_D(x) \right\}. \end{aligned}$$

The first equation comes from the definition of $D^{(j)}$ ($j \in \mathcal{J}$), the second equation from $\mathrm{Fix}(P_{Q^{(j)}}) = \{y \in \mathbb{R}^M : \|P_{Q^{(j)}}(y) - y\| = 0\} = Q^{(j)}$ ($j \in \mathcal{J}$), and the third and fourth equations from the definition of f_D (see (1.2)), $C = \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)}) \subset \mathbb{R}^N$, and $C \cap D \neq \emptyset$. This means that the solution set of problem (1.3) when $C \cap D \neq \emptyset$ coincides with the solution set of MSFP (1.1); i.e., the solution set of problem (1.3) is a generalization of the solution set of MSFP (1.1).

Let us define a general problem which includes problem (1.3). Since we can use nonexpansive mappings $T^{(i)} \colon \mathbb{R}^N \to \mathbb{R}^N$ ($i \in \mathcal{I}$) satisfying $\bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)}) \neq \emptyset$, we can also use $T \colon \mathbb{R}^N \to \mathbb{R}^N$ defined as follows: given $(v^{(i)})_{i\in\mathcal{I}} \subset (0,1)$ with $\sum_{i\in\mathcal{I}} v^{(i)} = 1$,

$$T := \sum_{i\in\mathcal{I}} v^{(i)} T^{(i)}.$$

The mapping T satisfies the nonexpansivity condition [29, Fact 2.1(b)], and the following holds [29, Fact 2.1(c)]:

$$\mathrm{Fix}(T) = \bigcap_{i\in\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right) \neq \emptyset.$$

Therefore, we can regard the constrained set in problem (1.3) as the fixed point set of a certain nonexpansive mapping (see [29, Fact 2.1] for other compositions of T such that $\mathrm{Fix}(T) = \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$). The function

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and $\|x - P_C(x)\| = \inf_{y\in C} \|x - y\|$ ($x \in \mathbb{R}^N$). Assumption (II) means that $P_{Q^{(j)}}$ can be easily computed within a finite number of arithmetic operations. $P_{Q^{(j)}}$ ($j \in \mathcal{J}$) satisfies the nonexpansivity condition [2, Proposition 2.10].

$f_D\colon \mathbb{R}^N \to \mathbb{R}$ defined by (1.2) is convex because A is linear and $\|\cdot\|^2$ is convex. Moreover, the gradient of f_D defined by

$$\nabla f_D(x) = {}^tAAx - \sum_{j\in\mathcal{J}} w^{(j)}\,{}^tA\left[P_{Q^{(j)}}(Ax)\right] \quad \left(x \in \mathbb{R}^N\right)$$

is Lipschitz continuous [6, Theorem 2(i)] with a constant $\rho({}^tAA)\sum_{j\in\mathcal{J}} w^{(j)}$,[5] where $\rho({}^tAA)$ is the spectral radius of ${}^tAA$ and ${}^tA$ stands for the transpose of A. Hence, we can regard the objective function in problem (1.3) as a differentiable, convex function with the Lipschitz continuous gradient. Now, we can formulate the following problem, which includes problem (1.3).

Problem 1.1. *Suppose that $T\colon \mathbb{R}^N \to \mathbb{R}^N$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, $f\colon \mathbb{R}^N \to \mathbb{R}$ is convex and differentiable, and $\nabla f\colon \mathbb{R}^N \to \mathbb{R}^N$ is L-Lipschitz continuous. Our objective is to*

$$\text{minimize } f(x) \text{ subject to } x \in \mathrm{Fix}(T).$$

There are many useful algorithms for minimizing a smooth (convex) function with a Lipschitz continuous gradient over the intersection of convex sets [11, 12, 26] or the intersection of fixed point sets [9, 17, 19, 21, 22, 23, 29, 30]. This paper focuses on conventional fixed point optimization algorithms [9, 17, 21, 22, 29, 30] and presents an algorithm which not only minimizes the objective function quickly but also converges in the fixed point set quickly.

Iterative algorithms [9, 21, 17, 22, 29, 30] have been proposed to solve Problem 1.1 under the assumptions that T is nonexpansive, f is strongly convex,[6] and ∇f is Lipschitz continuous; these assumptions are stronger than the assumptions of Problem 1.1 considered in this paper. The strong convexity of f and the Lipschitz continuity of ∇f guarantee that there exists a unique solution to Problem 1.1. By using the uniqueness and existence of the solution, we can prove that the algorithms converge to the solution; i.e., the algorithms can solve Problem 1.1 when f is strongly convex (see [18, 20] for algorithms for solving Problem 1.1 when f is strictly convex that has a unique solution).

Meanwhile, it is not always true that Problem 1.1 has a unique solution because f in Problem 1.1 and f_D defined by (1.2) do not satisfy the strong convexity condition. The main objective of this paper is to devise an iterative algorithm for solving Problem 1.1 with a convex objective function in contrast with the previous algorithms [9, 17, 18, 20, 21, 22, 29, 30]. The new result presented here is that the proposed algorithm (Algorithm 2.1) can solve problem (1.3), that is, find a compromise solution of IMSFP, which cannot be solved with the previous algorithms.

It is particularly worth noting that the algorithm, with *three-term conjugate gradient directions* [8, 17, 21, 24, 32, 33, 34], converges to a solution to Problem 1.1 faster than the conventional algorithms [4, 6, 28, 31, 35, 36] using the steepest descent direction of the objective function. Let us consider the following iterative algorithm for solving the problem of minimizing f over $\mathbb{R}^N$ (i.e., Problem 1.1 when T is the identity mapping, denoted by Id).

$$x_{n+1} := x_n + \alpha_n d_n^f \quad (n \in \mathbb{N}), \tag{1.4}$$

where x_n $(\in \mathbb{R}^N)$ is the nth approximation, α_n (> 0) is the step size, and d_n^f $(\in \mathbb{R}^N)$ is the search direction. The three-term conjugate gradient direction (TCGD) of f at x_{n+1} is

$$d_{n+1}^f := -\nabla f(x_{n+1}) + \delta_n^{(1)} d_n^f - \delta_n^{(2)} z_n, \tag{1.5}$$

where $(\delta_n^{(i)})_{n\in\mathbb{N}}$ $(\subset [0,\infty))$ $(i = 1, 2)$ and z_n $(\in \mathbb{R}^N)$ is an arbitrary point.

Algorithm (1.4) with a direction (1.5) when $\delta_n^{(i)} := 0$ $(i = 1, 2, n \in \mathbb{N})$, i.e., $x_{n+1} = x_n - \alpha_n \nabla f(x_n)$, is the steepest descent method. It satisfies $\langle d_n, \nabla f(x_n)\rangle < 0$ $(n \in \mathbb{N})$, called the *descent condition.* Since the methods satisfying the descent condition strictly decrease f at each iteration, they are powerfully useful to solve the problem of minimizing f over $\mathbb{R}^N$. However, the steepest descent method has a slow rate of convergence. Its acceleration has been of great interest. Much research in this direction covers, for example, the conjugate gradient methods and the three-term conjugate gradient method. Direction (1.5) when $\delta_n^{(2)} := 0$ $(n \in \mathbb{N})$; i.e., $d_{n+1}^f := -\nabla f(x_{n+1}) + \delta_n^{(1)} d_n^f$, is called the *conjugate gradient direction* [25, Chapter 5], and algorithm (1.4) with this direction is called the *conjugate gradient method.* Well-known formulas for $\delta_n^{(1)}$ have been proposed, including the Fletcher–Reeves, Polak–Ribiére–Polyak, Hestenes–Stiefel, and Dai–Yuan formulas (see [25, Chapter 5] for the definitions of their formulas). The conjugate gradient methods do not always satisfy the descent condition, and $\delta_n^{(1)}$ must be set appropriately so as to satisfy the descent condition.

[5] $S\colon \mathbb{R}^N \to \mathbb{R}^N$ is said to be Lipschitz continuous with $L > 0$ (L-Lipschitz continuous) if $\|S(x) - S(y)\| \leq L\|x - y\|$ $(x, y \in \mathbb{R}^N)$.

[6] $f\colon \mathbb{R}^N \to \mathbb{R}$ is said to be strongly convex with $\alpha > 0$ (α-strongly convex) if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) - (1/2)\alpha\lambda(1-\lambda)\|x - y\|^2$ $(\lambda \in [0,1], x, y \in \mathbb{R}^N)$.

Meanwhile, the *three-term conjugate gradient method* (TCGM) satisfies the descent condition [24, Subsection 2.1] without depending on the choice of $\delta_n^{(1)}$. This is because the third term, $\delta_n^{(2)} z_n$, in direction (1.5) plays an important role in satisfying the descent condition. Therefore, we can conclude that TCGM is a good way to solve the problem of minimizing f over $\mathbb{R}^N$. Hence, in this paper, we will present the algorithm with direction (1.5), which is used to minimize f quickly, for solving Problem 1.1.

Next, let us define TCGD for finding a fixed point of T. Consider the problem of minimizing a convex, differentiable functional g, with the l-Lipschitz gradient, over $\mathbb{R}^N$. Here, we define $T_g := \mathrm{Id} - r\nabla g$, where $r \in (0, 2/l]$. The mapping T_g satisfies the nonexpansivity condition and $\mathrm{Fix}(T_g) = \{x \in \mathbb{R}^N : g(x) = \min_{y \in \mathbb{R}^N} g(y)\}$ [16, Proposition 2.3]. Accordingly, we can regard the problem of minimizing g over $\mathbb{R}^N$ as the fixed point problem for a nonexpansive mapping. The well-known algorithm for solving the fixed point problem is as follows [15, 27]: given $x_0 \in \mathbb{R}^N$,

$$x_{n+1} := \beta_n x_0 + (1 - \beta_n) T_g(x_n) \quad (n \in \mathbb{N}), \tag{1.6}$$

where $(\beta_n)_{n\in\mathbb{N}} \subset [0, 1]$ with $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$. Algorithm (1.6) converges to a fixed point of T_g. Algorithm (1.6) with $T_g := \mathrm{Id} - r\nabla g$ can be represented as

$$\begin{aligned} x_{n+1} &= \beta_n x_0 + (1 - \beta_n)\left[x_n - r\nabla g(x_n)\right] \\ &= \beta_n x_0 + (1 - \beta_n)\left[x_n + r\left\{\frac{T_g(x_n) - x_n}{r}\right\}\right], \end{aligned}$$

which implies that algorithm (1.6) has the steepest descent direction of g at x_n, i.e.,

$$d_{n+1}^{T_g} := -\nabla g(x_n) = \frac{T_g(x_n) - x_n}{r}.$$

Therefore, we can define TCGD for finding a fixed point of T by

$$d_{n+1}^{T_g} := \frac{T_g(x_n) - x_n}{r} + \beta_n^{(1)} d_n^{T_g} + \beta_n^{(2)} w_n, \tag{1.7}$$

where $(\beta_n^{(i)})_{n\in\mathbb{N}}$ $(\subset [0, \infty))$ $(i = 1, 2)$ and w_n $(\in \mathbb{R}^N)$ is an arbitrary point.

From the above discussion, we can devise an algorithm with TCGDs (1.5) and (1.7) for solving Problem 1.1 that not only minimizes the objective function quickly but also converges in the fixed point set quickly. Section 2 describes the proposed algorithm and presents its convergence analyses. Section 3 gives numerical examples for the inconsistent split feasibility problems and demonstrates the effectiveness and convergence of the algorithm. Section 4 concludes the paper.

2 Algorithm with TCGDs and Its Convergence Analyses

Let us describe the algorithm for solving Problem 1.1.

Algorithm 2.1.

Step 0. Take a closed convex set K $(\subset \mathbb{R}^N)$, $(\alpha_n)_{n\in\mathbb{N}}$, $(\gamma_n)_{n\in\mathbb{N}} \subset [0, 1]$, $(\beta_n^{(i)})_{n\in\mathbb{N}}$, $(\delta_n^{(i)})_{n\in\mathbb{N}} \subset [0, 1]$ $(i = 1, 2)$, $\mu \in (0, 1]$, *and choose* $x_0 \in \mathbb{R}^N$ *arbitrarily. Let* $d_0^f := -\nabla f(x_0)$, $y_0 := x_0 + \alpha_0 d_0^f$, $d_0^T := T(y_0) - y_0$, *and* $n := 0$.

Step 1. Compute y_n $(\in \mathbb{R}^N)$ *as*

$$y_n := P_K\left(x_n + \alpha_n d_n^f\right)$$

and update d_{n+1}^T $(\in \mathbb{R}^N)$ *by*

$$d_{n+1}^T := T(y_n) - y_n + \beta_n^{(1)} d_n^T + \beta_n^{(2)} w_n,$$

where w_n $(\in \mathbb{R}^N)$ *is an arbitrary point.*

Step 2. Compute x_{n+1} $(\in \mathbb{R}^N)$ *as*

$$\begin{aligned} \bar{x}_{n+1} &:= P_K\left(y_n + \mu d_{n+1}^T\right), \\ x_{n+1} &:= P_K\left(\gamma_n x_0 + (1 - \gamma_n)\bar{x}_{n+1}\right) \end{aligned}$$

and update d_{n+1}^f $(\in \mathbb{R}^N)$ *by*

$$d_{n+1}^f := -\nabla f(x_{n+1}) + \delta_n^{(1)} d_n^f - \delta_n^{(2)} z_n,$$

where z_n $(\in \mathbb{R}^N)$ *is an arbitrary point. Put* $n := n + 1$, *and go to Step 1.*

2.1 Convergence analysis of Algorithm 2.1 when $\mathrm{Fix}(T)$ is bounded

This subsection assumes the following.

Assumption 2.1.

(A1) *K ($\supset \mathrm{Fix}(T)$) is bounded.*

(A2) *$(w_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are bounded.*

(A3) *$(\alpha_n)_{n\in\mathbb{N}} \subset (0, 2/L]$, $(\gamma_n)_{n\in\mathbb{N}} \subset (0,1]$, $(\beta_n^{(i)})_{n\in\mathbb{N}}$, $(\delta_n^{(i)})_{n\in\mathbb{N}} \subset [0,1]$ $(i=1,2)$ satisfy*

$$\text{(C1)}\ \sum_{n=0}^{\infty} \gamma_n = \infty,\ \text{(C2)}\ \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \gamma_n = 0,$$

$$\text{(C3)}\ \lim_{n\to\infty} \frac{1}{\gamma_{n+1}} \left| \frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n} \right| = 0,\ \text{(C4)}\ \lim_{n\to\infty} \frac{1}{\alpha_{n+1}} \left| 1 - \frac{\gamma_n}{\gamma_{n+1}} \right| = 0,$$

$$\text{(C5)}\ \lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0,\ \text{(C6)}\ \frac{\alpha_n}{\alpha_{n+1}}, \frac{\gamma_n}{\gamma_{n+1}} \le \sigma \text{ for some } \sigma < \infty,$$

$$\text{(C7)}\ \beta_n^{(i)} \le \gamma_n^2 \ (i=1,2)\ ,\ \text{(C8)}\ \delta_n^{(i)} \le \gamma_{n+1}^2 \ (i=1,2).$$

Consider problem (1.3) when $i_0 \in \mathcal{I}$ exists such that $C^{(i_0)}$ is bounded. Then, we can set a bounded K ($\supset \mathrm{Fix}(T^{(i_0)}) = C^{(i_0)}$) onto which the metric projection is easily computed (e.g., K is a closed ball with a large enough radius). In this case, we have $K \supset \mathrm{Fix}(T^{(i_0)}) \supset \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)}) =: \mathrm{Fix}(T)$, where $T := \sum_{i\in\mathcal{I}} v^{(i)} T^{(i)}$ (see also section 1 for the definition of T and the details of problem (1.3)). This implies that Assumption (A1) holds when at least one of the $C^{(i)}$ is bounded.

We can define $w_n := T(y_n) - y_n$ and $z_n := \nabla f(x_{n+1})$ ($n \in \mathbb{N}$) by referring to [17, 21, 33]. As a result, the boundedness of K (Assumption (A1)) guarantees that $(w_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are bounded (see the proofs of Lemmas 2.1 and 2.2 for the details). Accordingly, Assumption (A2) holds when K is bounded, and $(w_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are defined by $w_n := T(y_n) - y_n$ and $z_n := \nabla f(x_{n+1})$ ($n \in \mathbb{N}$).

Examples of $(\alpha_n)_{n\in\mathbb{N}}$, $(\gamma_n)_{n\in\mathbb{N}}$, $(\beta_n^{(i)})_{n\in\mathbb{N}}$, and $(\delta_n^{(i)})_{n\in\mathbb{N}}$ $(i = 1, 2)$ satisfying Assumption (A3) are $\alpha_n := (2/L)(1/(n+1)^a)$, $\gamma_n := 1/(n+1)^b$, $\beta_n^{(i)} := 1/(n+1)^{2b}$, and $\delta_n^{(i)} := 1/(n+2)^{2b}$ $(i = 1, 2)$, where $a \in (0, 1/2)$ and $b \in (a, 1-a)$.

The following is a convergence analysis of Algorithm 2.1.

Theorem 2.1. *Under Assumption 2.1, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by Algorithm 2.1 satisfies*

$$\lim_{n\to\infty} d(x_n, X^\star) = 0,$$

where $X^\star$ is the solution set of Problem 1.1 and $d(x_n, X^\star) := \inf_{x^\star \in X^\star} \|x_n - x^\star\|$ ($n \in \mathbb{N}$).

2.2 Proof of Theorem 2.1

We first prove the following lemmas.

Lemma 2.1. *$(y_n)_{n\in\mathbb{N}}$, $(T(y_n))_{n\in\mathbb{N}}$, and $(d_n^T)_{n\in\mathbb{N}}$ are bounded.*

Proof. Since K is bounded, $(y_n)_{n\in\mathbb{N}}$ ($\subset K$) is bounded. The nonexpansivity of T implies that $\|T(y_n) - T(x)\| \le \|y_n - x\|$ ($x \in \mathbb{R}^N$), and hence, $(T(y_n))_{n\in\mathbb{N}}$ is also bounded.

From (C2) and (C7), we have that $\lim_{n\to\infty} \beta_n^{(i)} = 0$ $(i = 1, 2)$. Accordingly, there exists $n_0 \in \mathbb{N}$ such that $\beta_n^{(1)} \le 1/3$ and $\beta_n^{(2)} \le 1$ for all $n \ge n_0$. Assumption (A2) and the boundedness of $(T(y_n))_{n\in\mathbb{N}}$ ensure that $M_1 := \max\{\sup\{\|T(y_n) - y_n\| \colon n \in \mathbb{N}\}, \sup\{\|w_n\| \colon n \in \mathbb{N}\}\} < \infty$ and $M_2 := \max\{M_1, \|d_{n_0}^T\|\} < \infty$. Hence, we have that $\|d_{n_0}^T\| \le 3M_2$. Suppose that $\|d_m^T\| \le 3M_2$ for some $m \ge n_0$. Then, $\|d_{m+1}^T\| = \|T(y_m) - y_m + \beta_m^{(1)} d_m^T + \beta_m^{(2)} w_m\| \le \|T(y_m) - y_m\| + \beta_m^{(1)} \|d_m^T\| + \beta_m^{(2)} \|w_m\| \le M_2 + (1/3)3M_2 + M_2 = 3M_2$. Therefore, induction shows that $\|d_n^T\| \le 3M_2$ ($n \ge n_0$); i.e., $(d_n^T)_{n\in\mathbb{N}}$ is bounded. □

Lemma 2.2. *$(\bar{x}_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, $(\nabla f(x_n))_{n\in\mathbb{N}}$, and $(d_n^f)_{n\in\mathbb{N}}$ are bounded.*

Proof. The boundedness of K guarantees that $(\bar{x}_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are bounded. From the Lipschitz continuity of ∇f, we have that $\|\nabla f(x_n) - \nabla f(x)\| \le L\|x_n - x\|$ ($x \in \mathbb{R}^N$). Hence, $(\nabla f(x_n))_{n\in\mathbb{N}}$ is bounded. Moreover, (C2) and (C8) imply that $\lim_{n\to\infty} \delta_n^{(i)} = 0$ $(i = 1, 2)$. Accordingly, a discussion similar to the proof of the boundedness of $(d_n^T)_{n\in\mathbb{N}}$ leads us to conclude that $(d_n^f)_{n\in\mathbb{N}}$ is bounded. □

Next, we prove the following.

Lemma 2.3. $\lim_{n\to\infty} \|x_{n+1} - x_n\|/\alpha_n = 0$.

Proof. The definition of x_n $(n \in \mathbb{N})$ and the nonexpansivity of P_K mean that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|P_K(\gamma_n x_0 + (1-\gamma_n)\bar{x}_{n+1}) - P_K(\gamma_{n-1}x_0 + (1-\gamma_{n-1})\bar{x}_n)\| \\ &\le \|(\gamma_n x_0 + (1-\gamma_n)\bar{x}_{n+1}) - (\gamma_{n-1}x_0 + (1-\gamma_{n-1})\bar{x}_n)\| \\ &= \|(1-\gamma_n)(\bar{x}_{n+1} - \bar{x}_n) + (\gamma_n - \gamma_{n-1})(x_0 - \bar{x}_n)\|,\end{aligned}$$

which from the triangle inequality and $M_3 := \sup\{\|x_0 - \bar{x}_n\| : n \in \mathbb{N}\} < \infty$ implies that

$$\|x_{n+1} - x_n\| \le (1-\gamma_n)\|\bar{x}_{n+1} - \bar{x}_n\| + M_3 |\gamma_n - \gamma_{n-1}|. \tag{2.1}$$

Meanwhile, $\bar{x}_{n+1}$ $(n \in \mathbb{N})$ can be represented as

$$\begin{aligned}\bar{x}_{n+1} &= P_K\left(y_n + \mu d_{n+1}^T\right) \\ &= P_K\left(y_n + \mu\left(T(y_n) - y_n + \beta_n^{(1)} d_n^T + \beta_n^{(2)} w_n\right)\right) \\ &= P_K\left((1-\mu)y_n + \mu T(y_n) + \mu\left(\beta_n^{(1)} d_n^T + \beta_n^{(2)} w_n\right)\right) \\ &=: P_K\left(\hat{T}(y_n) + \mu t_n\right),\end{aligned}$$

where $\hat{T} := (1-\mu)\mathrm{Id} + \mu T$ and $t_n := \beta_n^{(1)} d_n^T + \beta_n^{(2)} w_n$ $(n \in \mathbb{N})$. Accordingly, we have from the nonexpansivity of P_K and the triangle inequality that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\bar{x}_{n+1} - \bar{x}_n\| &= \left\|P_K\left(\hat{T}(y_n) + \mu t_n\right) - P_K\left(\hat{T}(y_{n-1}) + \mu t_{n-1}\right)\right\| \\ &\le \left\|\left(\hat{T}(y_n) + \mu t_n\right) - \left(\hat{T}(y_{n-1}) + \mu t_{n-1}\right)\right\| \\ &= \left\|\left(\hat{T}(y_n) - \hat{T}(y_{n-1})\right) + \mu(t_n - t_{n-1})\right\| \\ &\le \left\|\hat{T}(y_n) - \hat{T}(y_{n-1})\right\| + \mu\|t_n - t_{n-1}\|.\end{aligned}$$

Since T is nonexpansive, $\hat{T} := (1-\mu)\mathrm{Id} + \mu T$ is also nonexpansive. Hence, for all $n \in \mathbb{N}$,

$$\|\bar{x}_{n+1} - \bar{x}_n\| \le \|y_n - y_{n-1}\| + \mu\|t_n - t_{n-1}\|. \tag{2.2}$$

The definitions of y_n and d_n^f $(n \in \mathbb{N})$ and the nonexpansivity of P_K ensure that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|y_n - y_{n-1}\| &= \left\|P_K\left(x_n + \alpha_n d_n^f\right) - P_K\left(x_{n-1} + \alpha_{n-1} d_{n-1}^f\right)\right\| \\ &\le \left\|\left(x_n + \alpha_n d_n^f\right) - \left(x_{n-1} + \alpha_{n-1} d_{n-1}^f\right)\right\| \\ &= \Big\| \left[x_n + \alpha_n\left(-\nabla f(x_n) + \delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1}\right)\right] \\ &\quad - \left[x_{n-1} + \alpha_{n-1}\left(-\nabla f(x_{n-1}) + \delta_{n-2}^{(1)} d_{n-2}^f - \delta_{n-2}^{(2)} z_{n-2}\right)\right] \Big\| \\ &= \Big\| (x_n - \alpha_n \nabla f(x_n)) - (x_{n-1} - \alpha_n \nabla f(x_{n-1})) + (\alpha_{n-1} - \alpha_n)\nabla f(x_{n-1}) \\ &\quad + \alpha_n\left(\delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1}\right) - \alpha_{n-1}\left(\delta_{n-2}^{(1)} d_{n-2}^f - \delta_{n-2}^{(2)} z_{n-2}\right) \Big\|.\end{aligned}$$

Hence, the triangle inequality guarantees that

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \|(x_n - \alpha_n \nabla f(x_n)) - (x_{n-1} - \alpha_n \nabla f(x_{n-1}))\| + M_4|\alpha_{n-1} - \alpha_n| \\ &\quad + \alpha_n\left\|\delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1}\right\| + \alpha_{n-1}\left\|\delta_{n-2}^{(1)} d_{n-2}^f - \delta_{n-2}^{(2)} z_{n-2}\right\|,\end{aligned}$$

where $M_4 := \sup\{\|\nabla f(x_n)\| : n \in \mathbb{N}\} < \infty$. Since the mapping, $\mathrm{Id} - \alpha\nabla f$, where $\alpha \in (0, 2/L]$, satisfies the nonexpansivity condition [18, Proposition 2.3], we have, for all $n \in \mathbb{N}$,

$$\|(x_n - \alpha_n \nabla f(x_n)) - (x_{n-1} - \alpha_n \nabla f(x_{n-1}))\| \le \|x_n - x_{n-1}\|.$$

Moreover, the triangle inequality and (C8) imply that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1}\right\| &\le \delta_{n-1}^{(1)}\left\|d_{n-1}^f\right\| + \delta_{n-1}^{(2)}\|z_{n-1}\| \le M_5 \gamma_n^2, \\ \left\|\delta_{n-2}^{(1)} d_{n-2}^f - \delta_{n-2}^{(2)} z_{n-2}\right\| &\le \delta_{n-2}^{(1)}\left\|d_{n-2}^f\right\| + \delta_{n-2}^{(2)}\|z_{n-2}\| \le M_5 \gamma_{n-1}^2,\end{aligned}$$

where $M_5 := \max\{\sup\{\|d_n^f\| \colon n \in \mathbb{N}\}, \sup\{\|z_n\| \colon n \in \mathbb{N}\}\} < \infty$. Therefore, we find that, for all $n \in \mathbb{N}$,

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + M_4 |\alpha_{n-1} - \alpha_n| + M_5 \left(\alpha_n \gamma_n^2 + \alpha_{n-1}\gamma_{n-1}^2\right). \tag{2.3}$$

Since we have from (C7), for all $n \in \mathbb{N}$,

$$\|t_n\| = \left\|\beta_n^{(1)} d_n^T + \beta_n^{(2)} w_n\right\| \leq \beta_n^{(1)} \left\|d_n^T\right\| + \beta_n^{(2)} \|w_n\| \leq M_6 \gamma_n^2,$$

where $M_6 := \max\{\sup\{\|d_n^T\| \colon n \in \mathbb{N}\}, \sup\{\|w_n\| \colon n \in \mathbb{N}\}\} < \infty$, we find that

$$\|t_n - t_{n-1}\| \leq \|t_n\| + \|t_{n-1}\| \leq M_6 \left(\gamma_n^2 + \gamma_{n-1}^2\right). \tag{2.4}$$

Therefore, (2.1), (2.2), (2.3), and (2.4) guarantee that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1-\gamma_n) \|\bar{x}_{n+1} - \bar{x}_n\| + M_3 |\gamma_n - \gamma_{n-1}| \\
&\leq (1-\gamma_n) \{\|y_n - y_{n-1}\| + \mu \|t_n - t_{n-1}\|\} + M_3 |\gamma_n - \gamma_{n-1}| \\
&\leq (1-\gamma_n) \|x_n - x_{n-1}\| + M_3 |\gamma_n - \gamma_{n-1}| + M_4 |\alpha_{n-1} - \alpha_n| \\
&\quad + M_5 \left(\alpha_n \gamma_n^2 + \alpha_{n-1}\gamma_{n-1}^2\right) + M_6 \left(\gamma_n^2 + \gamma_{n-1}^2\right).
\end{aligned}$$

Accordingly, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\frac{\|x_{n+1} - x_n\|}{\alpha_n} &\leq (1-\gamma_n) \frac{\|x_n - x_{n-1}\|}{\alpha_n} + M_3 \frac{|\gamma_n - \gamma_{n-1}|}{\alpha_n} + M_4 \frac{|\alpha_{n-1} - \alpha_n|}{\alpha_n} \\
&\quad + M_5 \left(\gamma_n^2 + \frac{\alpha_{n-1}}{\alpha_n}\gamma_{n-1}^2\right) + M_6 \frac{\gamma_n^2 + \gamma_{n-1}^2}{\alpha_n} \\
&= (1-\gamma_n) \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + M_3 \frac{|\gamma_n - \gamma_{n-1}|}{\alpha_n} + M_4 \frac{|\alpha_{n-1} - \alpha_n|}{\alpha_n} \\
&\quad + M_5 \left(\gamma_n^2 + \frac{\alpha_{n-1}}{\alpha_n}\gamma_{n-1}^2\right) + M_6 \frac{\gamma_n^2 + \gamma_{n-1}^2}{\alpha_n} \\
&\quad + (1-\gamma_n) \left\{\frac{\|x_n - x_{n-1}\|}{\alpha_n} - \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}}\right\} \\
&\leq (1-\gamma_n) \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + M_3 \frac{|\gamma_n - \gamma_{n-1}|}{\alpha_n} + M_4 \frac{|\alpha_{n-1} - \alpha_n|}{\alpha_n} \\
&\quad + M_5 \left(\gamma_n^2 + \frac{\alpha_{n-1}}{\alpha_n}\gamma_{n-1}^2\right) + M_6 \frac{\gamma_n^2 + \gamma_{n-1}^2}{\alpha_n} + M_7 \left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right|,
\end{aligned}$$

where $M_7 := \sup\{\|x_{n+1} - x_n\| \colon n \in \mathbb{N}\} < \infty$. On the other hand, we have

$$\begin{aligned}
M_3 \frac{|\gamma_n - \gamma_{n-1}|}{\alpha_n} &= M_3 \gamma_n \frac{1}{\alpha_n} \frac{|\gamma_n - \gamma_{n-1}|}{\gamma_n} = M_3 \gamma_n \frac{1}{\alpha_n} \left|1 - \frac{\gamma_{n-1}}{\gamma_n}\right|, \\
M_7 \left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right| &= M_7 \gamma_n \frac{1}{\gamma_n} \left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right|.
\end{aligned}$$

We also have from $\alpha_{n-1} \leq 2/L$ that

$$\begin{aligned}
M_4 \frac{|\alpha_{n-1} - \alpha_n|}{\alpha_n} &= \frac{2}{L} M_4 \gamma_n \frac{1}{\gamma_n} \frac{|\alpha_{n-1} - \alpha_n|}{\frac{2}{L}\alpha_n} \leq \frac{2}{L} M_4 \gamma_n \frac{1}{\gamma_n} \frac{|\alpha_{n-1} - \alpha_n|}{\alpha_{n-1}\alpha_n} \\
&\leq \frac{2}{L} M_4 \gamma_n \frac{1}{\gamma_n} \left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right|.
\end{aligned}$$

Condition (C6) implies that

$$\begin{aligned}
M_5 \left(\gamma_n^2 + \frac{\alpha_{n-1}}{\alpha_n}\gamma_{n-1}^2\right) &\leq M_5 \left(\gamma_n^2 + \sigma \gamma_{n-1}^2\right) = M_5 \gamma_n \left(\gamma_n + \sigma \frac{\gamma_{n-1}}{\gamma_n}\gamma_{n-1}\right) \\
&\leq M_5 \gamma_n \left(\gamma_n + \sigma^2 \gamma_{n-1}\right), \\
M_6 \frac{\gamma_n^2 + \gamma_{n-1}^2}{\alpha_n} &= M_6 \gamma_n \left(\frac{\gamma_n}{\alpha_n} + \frac{\gamma_{n-1}^2}{\gamma_n \alpha_n}\right) = M_6 \gamma_n \left(\frac{\gamma_n}{\alpha_n} + \frac{\gamma_{n-1}}{\gamma_n}\frac{\gamma_{n-1}}{\gamma_n}\frac{\gamma_n}{\alpha_n}\right) \\
&\leq M_6 \gamma_n \left(\frac{\gamma_n}{\alpha_n} + \sigma^2 \frac{\gamma_n}{\alpha_n}\right).
\end{aligned}$$

Therefore, we find that, for all $n \in \mathbb{N}$,

$$\frac{\|x_{n+1} - x_n\|}{\alpha_n} \leq (1-\gamma_n) \frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + \gamma_n \Gamma_n,$$

where

$$\begin{aligned}\Gamma_n := & M_3 \frac{1}{\alpha_n} \left|1 - \frac{\gamma_{n-1}}{\gamma_n}\right| + \frac{2}{L} M_4 \frac{1}{\gamma_n} \left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right| + M_5 \left(\gamma_n + \sigma^2 \gamma_{n-1}\right) \\ & + M_6 \left(\frac{\gamma_n}{\alpha_n} + \sigma^2 \frac{\gamma_n}{\alpha_n}\right) + M_7 \frac{1}{\gamma_n} \left|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}\right|.\end{aligned}$$

Conditions (C2), (C3), (C4), and (C5) guarantee that $\lim_{n\to\infty} \Gamma_n = 0$. Hence, (C1) and [3, Lemma 1.2][7] lead us to

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\alpha_n} = 0.$$

This proves Lemma 2.3. □

Lemma 2.3 leads us to the following.

Lemma 2.4. $\lim_{n\to\infty} \|x_n - \bar{x}_{n+1}\|/\alpha_n = 0$ *and* $\lim_{n\to\infty} \|x_n - \bar{x}_{n+1}\| = 0$.

Proof. The nonexpansivity of P_K and $(\bar{x}_n)_{n\in\mathbb{N}} \subset K = \mathrm{Fix}(P_K)$ ensure that, for all $n \in \mathbb{N}$, $\|x_{n+1} - \bar{x}_{n+1}\| = \|P_K(\gamma_n x_0 + (1-\gamma_n)\bar{x}_{n+1}) - P_K(\bar{x}_{n+1})\| \leq \|(\gamma_n x_0 + (1-\gamma_n)\bar{x}_{n+1}) - \bar{x}_{n+1}\| = \gamma_n \|x_0 - \bar{x}_{n+1}\|$. Accordingly, the triangle inequality implies that, for all $n \in \mathbb{N}$, $\|x_n - \bar{x}_{n+1}\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - \bar{x}_{n+1}\| \leq \|x_n - x_{n+1}\| + \gamma_n \|x_0 - \bar{x}_{n+1}\|$, which means that

$$\frac{\|x_n - \bar{x}_{n+1}\|}{\alpha_n} \leq \frac{\|x_n - x_{n+1}\|}{\alpha_n} + \frac{\gamma_n}{\alpha_n} \|x_0 - \bar{x}_{n+1}\|.$$

The boundedness of $(\bar{x}_n)_{n\in\mathbb{N}}$, Lemma 2.3, and (C5) guarantee that $\lim_{n\to\infty} \|x_n - \bar{x}_{n+1}\|/\alpha_n = 0$. Moreover, from (C2), we find that $\lim_{n\to\infty} \|x_n - \bar{x}_{n+1}\| = 0$. □

Lemma 2.5. $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Proof. From the triangle inequality, we have, for all $n \in \mathbb{N}$, $\|x_n - P_K(\hat{T}(x_n))\| \leq \|x_n - \bar{x}_{n+1}\| + \|\bar{x}_{n+1} - P_K(\hat{T}(x_n))\|$. Meanwhile, $\bar{x}_{n+1} = P_K(\hat{T}(y_n) + \mu t_n)$ $(n \in \mathbb{N})$ and the nonexpansivity of P_K and $\hat{T}$ imply that, for all $n \in \mathbb{N}$, $\|\bar{x}_{n+1} - P_K(\hat{T}(x_n))\| = \|P_K(\hat{T}(y_n) + \mu t_n) - P_K(\hat{T}(x_n))\| \leq \|\hat{T}(y_n) - \hat{T}(x_n) + \mu t_n\| \leq \|y_n - x_n\| + \mu \|t_n\|$. Moreover,

$$\begin{aligned}\|y_n - x_n\| &= \left\|P_K\left(x_n + \alpha_n d_n^f\right) - P_K(x_n)\right\| \leq \left\|\left(x_n + \alpha_n d_n^f\right) - x_n\right\| \leq M_5 \alpha_n, \\ \|t_n\| &= \left\|\beta_n^{(1)} d_n^T + \beta_n^{(2)} w_n\right\| \leq \beta_n^{(1)} \left\|d_n^T\right\| + \beta_n^{(2)} \|w_n\| \leq M_6 \gamma_n^2,\end{aligned}$$

where $M_5 := \max\{\sup\{\|d_n^f\| \colon n \in \mathbb{N}\}, \sup\{\|z_n\| \colon n \in \mathbb{N}\}\} < \infty$ and $M_6 := \max\{\sup\{\|d_n^T\| \colon n \in \mathbb{N}\}, \sup\{\|w_n\| \colon n \in \mathbb{N}\}\} < \infty$. Hence, for all $n \in \mathbb{N}$,

$$\left\|x_n - P_K\left(\hat{T}(x_n)\right)\right\| \leq \|x_n - \bar{x}_{n+1}\| + M_5 \alpha_n + M_6 \gamma_n^2.$$

Lemma 2.4 and (C2) ensure that $\lim_{n\to\infty} \|x_n - P_K(\hat{T}(x_n))\| = 0$. Since $\mathrm{Fix}(\hat{T}) = \mathrm{Fix}(T) \subset K$, we find that $\lim_{n\to\infty} \|x_n - \hat{T}(x_n)\| = 0$ [1, Theorems 3.7 and 3.9].[8] From $\hat{T} = (1-\mu)\mathrm{Id} + \mu T$, we have $0 = \lim_{n\to\infty} \|x_n - \hat{T}(x_n)\| = \mu \lim_{n\to\infty} \|x_n - T(x_n)\|$; i.e., $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. □

We can also prove the following.

Lemma 2.6. $0 \leq N_1 \|\bar{x}_{n+1} - x_n\|/\alpha_n + 2\langle x - x_n, \nabla f(x_n)\rangle + N_2 \alpha_n + N_3 \gamma_n/\alpha_n$ $(x \in \mathrm{Fix(T)}, n \in \mathbb{N})$*, where* $\langle \cdot, \cdot \rangle$ *stands for the inner product of* $\mathbb{R}^N$*, and* N_i $(i = 1,2,3)$ *are positive constants.*

[7] Lemma 1.2 in [3] is as follows: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers defined by $x_{n+1} \leq (1-a_n)x_n + a_n b_n$ $(n \in \mathbb{N})$, where $(a_n)_{n\in\mathbb{N}}$ $(\subset [0,1])$ with $\sum_{n=0}^{\infty} a_n = \infty$ and $(b_n)_{n\in\mathbb{N}}$ with $b_n \geq 0$ and $\lim_{n\to\infty} b_n = 0$. Then, $\lim_{n\to\infty} x_n = 0$.

[8] Theorems 3.7 and 3.9 in [1] lead us to the following: Suppose that P_K is the metric projection onto a closed convex K, $T \colon \mathbb{R}^N \to \mathbb{R}^N$ is nonexpansive with $K \cap \mathrm{Fix}(T) \neq \emptyset$, and $(x_n)_{n\in\mathbb{N}}$ is bounded. Then, $\lim_{n\to\infty} \|x_n - P_K(T(x_n))\| = 0$ if and only if $\lim_{n\to\infty} \|x_n - P_K(x_n)\| = 0$ and $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Proof. Choose $x \in \mathrm{Fix}(T)$ arbitrarily. Then, we have from $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T}) \subset K = \mathrm{Fix}(P_K)$ that $x = P_K(x)$ and $x = \hat{T}(x)$. The nonexpansivity of P_K and $\bar{x}_{n+1} = P_K(\hat{T}(y_n) + \mu t_n)$ $(n \in \mathbb{N})$ mean that, for all $n \in \mathbb{N}$, $\|\bar{x}_{n+1} - x\|^2 = \|P_K(\hat{T}(y_n) + \mu t_n) - P_K(x)\|^2 \leq \|(\hat{T}(y_n) - \hat{T}(x)) + \mu t_n\|^2$. Hence, the inequality, $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ $(x, y \in \mathbb{R}^N)$, guarantees that, for all $n \in \mathbb{N}$,

$$\|\bar{x}_{n+1} - x\|^2 \leq \left\|\hat{T}(y_n) - \hat{T}(x)\right\|^2 + 2\mu \left\langle t_n, \hat{T}(y_n) - \hat{T}(x) + \mu t_n \right\rangle,$$

which implies that

$$\begin{aligned}
\|\bar{x}_{n+1} - x\|^2 &\leq \|y_n - x\|^2 + 2\mu \|t_n\| \left\|\hat{T}(y_n) - \hat{T}(x) + \mu t_n\right\| \\
&\leq \|y_n - x\|^2 + 2\mu M_6 \gamma_n^2 \left\|\hat{T}(y_n) - \hat{T}(x) + \mu t_n\right\| \\
&\leq \|y_n - x\|^2 + M_8 \gamma_n^2.
\end{aligned}$$

The first inequality comes from the nonexpansivity of $\hat{T}$ and the Cauchy-Schwarz inequality, the second inequality from $\|t_n\| \leq M_6 \gamma_n^2$ $(n \in \mathbb{N})$, and the third inequality from $M_8 := \sup\{2\mu M_6 \|\hat{T}(y_n) - \hat{T}(x) + \mu t_n\| \colon n \in \mathbb{N}\} < \infty$. Moreover, we find from the equation, $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ $(x, y \in \mathbb{R}^N)$, that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|y_n - x\|^2 &= \left\|P_K\left(x_n + \alpha_n d_n^f\right) - P_K(x)\right\|^2 \\
&\leq \left\|(x_n - x) + \alpha_n d_n^f\right\|^2 \\
&= \|x_n - x\|^2 + 2\alpha_n \left\langle x_n - x, d_n^f \right\rangle + \alpha_n^2 \left\|d_n^f\right\|^2 \\
&\leq \|x_n - x\|^2 + 2\alpha_n \left\langle x_n - x, d_n^f \right\rangle + M_5^2 \alpha_n^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_n - x\|^2 &\leq \|x_n - x\|^2 + 2\alpha_n \left\langle x_n - x, -\nabla f(x_n) + \delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1} \right\rangle + M_5^2 \alpha_n^2 \\
&= \|x_n - x\|^2 + 2\alpha_n \langle x_n - x, -\nabla f(x_n)\rangle + M_5^2 \alpha_n^2 \\
&\quad + 2\alpha_n \left\langle x_n - x, \delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1} \right\rangle \\
&\leq \|x_n - x\|^2 + 2\alpha_n \langle x - x_n, \nabla f(x_n)\rangle + M_5^2 \alpha_n^2 \\
&\quad + 2\alpha_n \|x_n - x\| \left\|\delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1}\right\| \\
&\leq \|x_n - x\|^2 + 2\alpha_n \langle x - x_n, \nabla f(x_n)\rangle + M_5^2 \alpha_n^2 + 2\alpha_n \|x_n - x\| M_5 \gamma_n^2.
\end{aligned}$$

The first inequality comes from the definition of d_n^f $(n \in \mathbb{N})$, the second inequality from the Cauchy-Schwarz inequality, and the third inequality from $\|\delta_{n-1}^{(1)} d_{n-1}^f - \delta_{n-1}^{(2)} z_{n-1}\| \leq M_5 \gamma_n^2$ $(n \in \mathbb{N})$. Therefore, for all $n \in \mathbb{N}$,

$$\|\bar{x}_{n+1} - x\|^2 \leq \|x_n - x\|^2 + 2\alpha_n \langle x - x_n, \nabla f(x_n)\rangle + M_5^2 \alpha_n^2 + M_9 \gamma_n^2,$$

where $M_9 := \sup\{2\alpha_n \|x_n - x\| M_5 + M_8 \colon n \in \mathbb{N}\} < \infty$. Accordingly, we have

$$\begin{aligned}
0 &\leq \|x_n - x\|^2 - \|\bar{x}_{n+1} - x\|^2 + 2\alpha_n \langle x - x_n, \nabla f(x_n)\rangle + M_5^2 \alpha_n^2 + M_9 \gamma_n^2 \\
&= (\|x_n - x\| + \|\bar{x}_{n+1} - x\|)(\|x_n - x\| - \|\bar{x}_{n+1} - x\|) + M_5^2 \alpha_n^2 + M_9 \gamma_n^2 \\
&\quad + 2\alpha_n \langle x - x_n, \nabla f(x_n)\rangle \\
&\leq (\|x_n - x\| + \|\bar{x}_{n+1} - x\|) \|x_n - \bar{x}_{n+1}\| + M_5^2 \alpha_n^2 + M_9 \gamma_n^2 \\
&\quad + 2\alpha_n \langle x - x_n, \nabla f(x_n)\rangle,
\end{aligned}$$

which implies that, for all $n \in \mathbb{N}$,

$$0 \leq M_{10} \frac{\|x_n - \bar{x}_{n+1}\|}{\alpha_n} + M_5^2 \alpha_n + M_9 \frac{\gamma_n}{\alpha_n} + 2\langle x - x_n, \nabla f(x_n)\rangle,$$

where $M_{10} := \sup\{\|x_n - x\| + \|\bar{x}_{n+1} - x\| : n \in \mathbb{N}\} < \infty$. This proves Lemma 2.6. □

We can prove Theorem 2.1 by using the above lemmas.

Proof of Theorem 2.1 We prove the claim by contradiction. Suppose on the contrary that there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $d(x_{n_k}, X^\star) \geq \varepsilon$ for some $\varepsilon > 0$. Since $(x_{n_k})_{k \in \mathbb{N}}$ is bounded, it has a limit point x^*. Hence, the continuity of T ensures that $(T(x_{n_k}))_{k\in\mathbb{N}}$ converges to $T(x^*)$. Lemma 2.5 guarantees that

$$0 = \lim_{k\to\infty} \|x_{n_k} - T(x_{n_k})\| = \|x^* - T(x^*)\|;$$

i.e., $x^* \in \mathrm{Fix}(T)$. Lemmas 2.4 and 2.6, (C2), (C5), and the continuity of ∇f lead us to

$$0 \leq \langle x - x^*, \nabla f(x^*)\rangle \quad (x \in \mathrm{Fix}(T)).$$

The convexity of f and the closedness and convexity of $\mathrm{Fix}(T)$ imply that x^* ($\in \mathrm{Fix}(T)$) satisfies the above inequality if and only if x^* ($\in \mathrm{Fix}(T)$) is a solution to Problem 1.1; i.e., $x^* \in X^\star$ [13, Proposition 2.1 in Chapter II]. Therefore, we conclude that $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x^* \in X^\star$. We also have from $x^* \in X^\star$ that $d(x_{n_k}, X^\star) = \inf_{x^\star\in X^\star} \|x_{n_k} - x^\star\| \leq \|x_{n_k} - x^*\|$. Since $(x_{n_k})_{k\in\mathbb{N}}$ converges to x^*, we find

$$0 < \varepsilon \leq \limsup_{k\to\infty} d(x_{n_k}, X^\star) \leq \limsup_{k\to\infty} \|x_{n_k} - x^*\| = \lim_{k\to\infty} \|x_{n_k} - x^*\| = 0,$$

which is a contradiction. Therefore, we can conclude that $(x_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} d(x_n, X^\star) = 0$. □

2.3 Convergence analysis of Algorithm 2.1 when $\mathrm{Fix}(T)$ is unbounded

Subsection 2.1 gave the convergence analysis of Algorithm 2.1 when $\mathrm{Fix}(T)$ is bounded (see Assumption (A1)). In this subsection, let us consider Algorithm 2.1 when $\mathrm{Fix}(T)$ is unbounded. We need to replace Assumption (A1) with the following so as to prove that $(d_n^T)_{n\in\mathbb{N}}$, $(d_n^f)_{n\in\mathbb{N}}$, and $(x_n)_{n\in\mathbb{N}}$ are bounded.

(A1)' $K = \mathbb{R}^N$, and $(y_n)_{n\in\mathbb{N}}$ and $(\nabla f(x_n))_{n\in\mathbb{N}}$ are bounded.

Suppose that Assumptions (A1)', (A2), and (A3) are satisfied. Since $(y_n)_{n\in\mathbb{N}}$ is bounded, and T is nonexpansive, $(T(y_n) - y_n)_{n\in\mathbb{N}}$ is bounded. Hence, we have from the proof of Lemma 2.1 that $(d_n^T)_{n\in\mathbb{N}}$ is bounded. Moreover, the boundedness of $(\nabla f(x_n))_{n\in\mathbb{N}}$ and the proofs of Lemmas 2.1 and 2.2 lead us to conclude that $(d_n^f)_{n\in\mathbb{N}}$ is bounded. The boundedness of $(d_n^T)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ means that $(\bar{x}_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are bounded. This implies that we can prove Lemmas 2.1 and 2.2 under Assumptions (A1)', (A2), and (A3). Therefore, we can also prove Lemmas 2.3, 2.4, 2.5, and 2.6 under Assumptions (A1)', (A2), and (A3) because the proofs of Lemmas 2.3, 2.4, 2.5, and 2.6 in subsection 2.2 use the boundedness of $(d_n^T)_{n\in\mathbb{N}}$, $(d_n^f)_{n\in\mathbb{N}}$, $(\bar{x}_n)_{n\in\mathbb{N}}$, and $(x_n)_{n\in\mathbb{N}}$, not the boundedness of $\mathrm{Fix}(T)$.

We can perform a convergence analysis of Algorithm 2.1 when $\mathrm{Fix}(T)$ is unbounded by referring to the proof of Theorem 2.1.

Theorem 2.2. *Under Assumptions (A1)', (A2), and (A3), the sequence $(x_n)_{n\in\mathbb{N}}$ generated by Algorithm 2.1 satisfies*

$$\lim_{n\to\infty} d(x_n, X^\star) = 0.$$

3 Numerical Examples

Let us see how Algorithm 2.1 works in solving a concrete IMSFP. We set A ($\in \mathbb{R}^{500\times 1000}$) and x_0 ($\in \mathbb{R}^{1000}$) given randomly by MATLAB, and closed balls $C^{(i)}$ ($\subset \mathbb{R}^{1000}$) ($i \in \mathcal{I} := \{1,2,3,4,5\}$) and $Q^{(j)}$ ($\subset \mathbb{R}^{500}$) ($j \in \mathcal{J} := \{1,2,3\}$) with $\bigcap_{i\in\mathcal{I}} C^{(i)} \neq \emptyset$ and $\bigcap_{i\in\mathcal{I}} C^{(i)} \cap \bigcap_{j\in\mathcal{J}} D^{(j)} = \emptyset$, where $D^{(j)} := \{x \in \mathbb{R}^{1000} \colon Ax \in Q^{(j)}\}$ ($j \in \mathcal{J}$). We used $T := (1/5)\sum_{i\in\mathcal{I}} P_{C^{(i)}}$ and $K := C^{(1)}$ with $K \supset \mathrm{Fix}(T) = \bigcap_{i\in\mathcal{I}} C^{(i)} \neq \emptyset$. We also used $\nabla f(x) := {}^t\!AAx - (1/3)\sum_{j\in\mathcal{J}} {}^t\!A[P_{Q^{(j)}}(Ax)]$ ($x \in \mathbb{R}^{1000}$), $w_n := T(y_n) - y_n$, and $z_n := \nabla f(x_{n+1})$ ($n \in \mathbb{N}$) (see subsection 2.1 for the setting of w_n and z_n). The step-size sequences in the experiment were $\alpha_n := 1/(n+1)^{0.4}$, $\gamma_n := 1/(n+1)^{0.5}$, $\beta_n^{(i)} := 1/(n+1)$, and $\delta_n^{(i)} := 1/(n+2)$ ($i = 1,2, n \in \mathbb{N}$). Since Assumption 2.1 holds, Algorithm 2.1 can find a minimizer of f over $\mathrm{Fix}(T)$, i.e., the compromise solution of IMSFP defined by $C^{(i)}$ ($i \in \mathcal{I}$), $Q^{(j)}$ ($j \in \mathcal{J}$), and A ($\in \mathbb{R}^{1000\times 500}$) (see section 1 and Theorem 2.1). The computer used in the experiment had an Intel Boxed Core i7 i7-870 2.93 GHz 8 M CPU and 8 GB of memory. The language was MATLAB 7.13.

We compared Algorithm 2.1 with the following algorithms: $d_0^f := -\nabla f(x_0)$,

$$y_n := P_K\left(x_n + \frac{10^{-3}}{(n+1)^{0.4}} d_n^f\right),$$
$$x_{n+1} := P_K\left(\frac{1}{(n+1)^{0.5}} x_0 + \left(1 - \frac{1}{(n+1)^{0.5}}\right) P_K(T(y_n))\right) \quad (n \in \mathbb{N}),$$

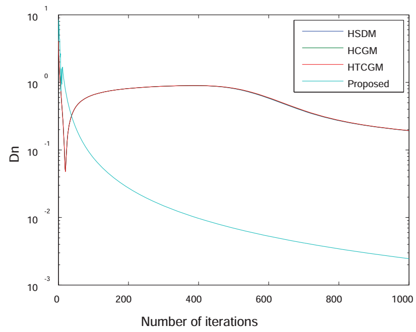

Figure 1: Behavior of $D_n := \|x_n - T(x_n)\|$ for HSDM, HCGM, HTCGM, and Algorithm 2.1 (Proposed)

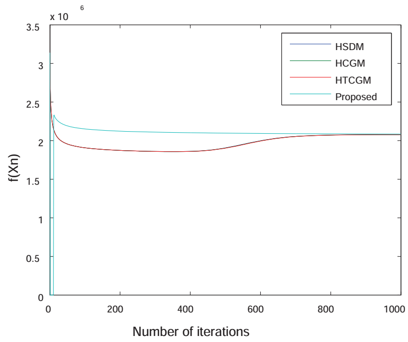

Figure 2: Behavior of $f(x_n) := (1/2)\sum_{j\in\mathcal{J}}(1/3)\|P_{Q^{(j)}}(Ax_n) - Ax_n\|^2$ for HSDM, HCGM, HTCGM, and Algorithm 2.1 (Proposed)

where d^f_{n+1} is defined by one of

$$d^f_{n+1} := -\nabla f(x_{n+1}), \tag{3.1}$$

$$d^f_{n+1} := -\nabla f(x_{n+1}) + \frac{1}{n+2} d^f_n, \tag{3.2}$$

$$d^f_{n+1} := -\nabla f(x_{n+1}) + \frac{1}{n+2} d^f_n - \frac{1}{n+2}\nabla f(x_{n+1}). \tag{3.3}$$

The algorithms with directions (3.1), (3.2), and (3.3) are referred to the hybrid steepest descent method (HSDM) [30], the hybrid conjugate gradient method (HCGM) [22], and the hybrid three-term conjugate gradient method (HTCGM) [17], respectively. We note that HTCGM coincides with Algorithm 2.1 when $\mu := 1$, and $\beta^{(i)}_n := 0$ $(i = 1, 2, n \in \mathbb{N})$. We executed Algorithm 2.1 (Proposed) generated by

$$\begin{aligned}
y_n &:= P_K\left(x_n + \frac{10^{-3}}{(n+1)^{0.4}} d^f_n\right),\\
d^T_{n+1} &:= T(y_n) - y_n + \frac{1}{n+1} d^T_n + \frac{1}{n+1}\left(T(y_n) - y_n\right),\\
\bar{x}_{n+1} &:= P_K\left(y_n + d^T_{n+1}\right),\\
x_{n+1} &:= P_K\left(\frac{1}{(n+1)^{0.5}} x_0 + \left(1 - \frac{1}{(n+1)^{0.5}}\right)\bar{x}_{n+1}\right),\\
d^f_{n+1} &:= -\nabla f(x_{n+1}) + \frac{1}{n+2} d^f_n - \frac{1}{n+2}\nabla f(x_{n+1}).
\end{aligned}$$

Note that Algorithm 2.1 has TCGD, which is used to find a fixed point of T quickly,

$$d^T_{n+1} := T(y_n) - y_n + \frac{1}{n+1} d^T_n + \frac{1}{n+1}\left(T(y_n) - y_n\right) \quad (n \in \mathbb{N}), \tag{3.4}$$

while HTCGM has the steepest descent direction for finding a fixed point of T (see section 1 for the details),

$$d^T_{n+1} := T(y_n) - y_n \quad (n \in \mathbb{N}). \tag{3.5}$$

Accordingly, we can expect that Algorithm 2.1 converges to a fixed point of T faster than HTCGM.

We define $D_n := \|x_n - T(x_n)\|$ $(n \in \mathbb{N})$. If $(D_n)_{n\in\mathbb{N}}$ converges to 0, $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of T. Figure 1 describes the behavior of D_n generated by HSDM, HCGM, HTCGM, and Algorithm 2.1. $(D_n)_{n\geq 500}$ generated by HSDM, HCGM, and HTCGM converge to 0, and their behaviors are the same. This is because their search directions for finding a fixed point of T are generated by (3.5). Moreover, we can see from Figure 1 that they slowly converge to a fixed point of T. This is because direction (3.5) is expressed as the steepest descent direction of a certain convex function (see also section 1). Meanwhile, $(D_n)_{n\in\mathbb{N}}$ generated by Algorithm 2.1 converges to 0 faster than $(D_n)_{n\in\mathbb{N}}$ generated by HSDM, HCGM, and

HTCGM. This implies that Algorithm 2.1 with TCGD (3.4) can find a fixed point of T faster than the algorithms with the steepest descent direction (3.5).

Finally, let us see the behavior of the objective function defined by $f(x) := (1/2)\sum_{j\in\mathcal{J}}(1/3)\|P_{Q(j)}(Ax)-Ax\|^2$ $(x \in \mathbb{R}^{1000})$. Figure 2 shows that $(f(x_n))_{n\in\mathbb{N}}$ generated by HSDM, HCGM, HTCGM, and Algorithm 2.1 converge. Since Figure 1 describes that Algorithm 2.1 decreases D_n faster than HSDM, HCGM, and HTCGM, we can conclude that Algorithm 2.1 is effective for solving Problem 1.1.

4 Conclusion

This paper discussed the inconsistent multiple-set split feasibility problem and proved that the compromise solution of the problem can be expressed as a minimizer of a convex objective function over the fixed point set of a nonexpansive mapping. We presented the algorithm, with the three-term conjugate gradient directions, for solving the convex minimization problem and provided its convergence analyses. The analyses guarantee that the algorithm, with slowly diminishing step-size sequences, converges to a solution to the convex minimization problem. Finally, we gave numerical results to support the convergence analyses on the algorithm. The numerical results showed that the proposed algorithm performs better than the algorithms with the steepest descent directions.

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