DISTRIBUTED CONVEX OPTIMIZATION ALGORITHMS AND THEIR APPLICATION TO DISTRIBUTED CONTROL IN PEER-TO-PEER DATA STORAGE SYSTEM

HIDEAKI IIDUKA

ABSTRACT. This paper introduces two incentive schemes to control peer-to-peer (P2P) data storage systems. One is a symmetric scheme that imposes a rule in which the contribution level of each peer (storage space offered by each peer) is equal to its use of the service (storage space used for storing its own data). The other is a profit-oriented pricing scheme that allows an operator, who manages the whole system, to buy and sell storage space from and to peers so as to maximize its profit. We show that the storage allocation problems entailed by these schemes can be formulated as convex minimization problems. We propose two optimization algorithms, based on distributed convex optimization techniques, for solving these problems. One algorithm works when all peers act on the basis of the symmetric scheme and finds an optimal storage allocation that maximizes a performance measure, called social welfare, in the whole system. The other algorithm works when the operator and all peers act on the basis of the profit-oriented pricing scheme and finds an optimal allocation that maximizes the weighted mean of the social welfare and the operator's profit. We give numerical results proving that the algorithms converge to the solutions to the storage allocation problems.

1. INTRODUCTION

Peer-to-Peer (P2P) network models have attracted a great deal of attention. The concept of the P2P network model is completely different from that of a conventional client-server network model. While a conventional server-client network model explicitly distinguishes hosts providing services (servers) from hosts receiving services (clients), a P2P network model does not assign fixed roles to hosts. Hosts composing P2P networks, referred to as *peers*, can be both servers and clients, and as a result, P2P networks function as autonomous, distributed systems.

In a P2P data storage system, each peer offers some of its memory capacity as a service to others and benefits from storing its own data on the system. An online storage service is valuable only if it gives users reliable

2010 *Mathematics Subject Classification.* 65K05, 68M10, 90C25, 90C90.

Key words and phrases. convex optimization, distributed optimization, fixed point set, incentives, nonexpansive mapping, peer-to-peer data storage system, strongly convex function.

access to their data. Hence, the system needs to cope with disk failures and with peers disconnecting their disk from the system. This implies that data replicates must be spread over several, reliable peers. Accordingly, the system requires peers to offer a sufficient part of their disk space to the system and remain online often enough. However, both of these requirements are costly to peers. Therefore, the system is at risk of collapse when peers do not offer enough storage capacity, i.e., when peers behave selfishly.

In this paper, we focus on *incentives* [21] to make peers contribute to P2P data storage systems. There are many incentive schemes that are applicable to the economics of P2P file sharing networks (e.g., wireless local area networks (WLANs) [13]), and they can be generally categorized as *symmetric or payment-based* scheme (see, e.g., [1, 13] and references therein). However, these schemes cannot be applied to P2P data storage systems because their economic implications are essentially incompatible.

To resolve this issue, reference [21] presented two incentive schemes to control P2P data storage systems. One is a *symmetric scheme* [21, subsection II.C 1)] based on the idea that every peer should contribute to the system in terms of service at least as much as it benefits from others. The other is a *profit-oriented pricing scheme* [21, subsection II.C 2)] based on monetary exchanges where peers can buy storage space and sell some of their disk capacity. Reference [21] analyzed whether it is socially better to impose the symmetric scheme or the profit-oriented pricing scheme. The performance measure is social welfare, defined by the sum of the utility functions of all peers and an operator, who manages the P2P data storage system.

The main objective of storage allocation in P2P systems is to find optimal storage capacities of all peers which maximize the social welfare as much as possible. The analyses in [21] assumed the existence of central authority to supervise the peers' behavior; i.e., the central authority knows the private information of all peers, such as the explicit forms of all peers' utility functions and strategies. In contrast to such a centralized system control, this paper discusses *distributed* system control for P2P systems. Our distributed mechanisms can be applied to any P2P network without a central authority (e.g., a pure P2P network such as Winny and Gnutella), and they enable each peer to find a maximizer of the social welfare without using the private information of other peers, such as their utility functions and strategy sets.

In this paper, we first show that the storage allocation problems (problems of maximizing the social welfare) caused by the two incentive schemes can be formulated as convex minimization problems over *the fixed point sets of nonexpansive mappings*. We then propose two distributed convex optimization algorithms, based on *fixed point theory* [2], [3, Chapter 4], [14, Chapter 3], [15, Chapter 1], for solving them.

A number of distributed convex optimization algorithms have been presented (see [7, Subchapter 8.2], [8, 9, 11, 12, 16, 18, 19, 20, 22, 23, 24, 25, 27, 30] and reference therein). However, the literature does not seem to have any algorithm for solving convex minimization problems over the fixed point sets. While the previously reported results in [21] presented useful mathematical models of P2P data storage systems, to our knowledge, there are no references on distributed control algorithms for controlling these systems.

Our main contribution is to devise distributed convex optimization techniques to solve storage allocation problems of P2P data storage systems through incentive schemes. We believe that our distributed approach is good for optimal control problems [10, 17, 26], network flow problems [5, 6, 20, 29], and resource allocation problems [28, Chapters 4 and 5]. This is because it can be applied to the more general problem of minimizing the sum of strongly convex objective functions over the intersection of fixed point sets of nonexpansive mappings. Therefore, we believe that the results in this paper will provide a glimpse into the inherent connection between distributed algorithms and control problems in networked systems.

This paper is organized as follows. Section 2 gives the mathematical preliminaries. Section 3 shows that the storage allocation problems entailed by the two incentive schemes (symmetric scheme and profit-oriented pricing scheme) can be formulated as convex minimization problems over the fixed point sets of certain nonexpansive mappings. Section 4 discusses the distributed control under the symmetric scheme. Section 5 discusses the distributed control under the profit-oriented pricing scheme. We show that the algorithms presented in sections 4 and 5 converge to the solutions to the storage allocation problems under realistic assumptions. Section 6 applies the algorithms to concrete storage allocation problems and provides numerical results showing they converge to the solutions. Section 7 concludes the paper.

2. Preliminaries

This section describes the basic model of a P2P data storage system studied in [21], which was the first study to propose incentive schemes for controlling P2P data storage systems.

Consider a P2P data storage system network in which peer i ($i \in \mathcal{I} := \{1, 2, \ldots, K\}$) *offers* a storage capacity $c_o^{(i)}$ that is to be shared with other peers and demands a storage capacity $c_s^{(i)}$ that is to be used for *storing* its own data.

The supply and demand functions of peer i are defined as follows: there exist $a^{(i)}, b^{(i)}, p_{\max}^{(i)}$ (> 0), and $p_{\min}^{(i)}$ (≥ 0) such that, for all $p \geq 0$,

$$s^{(i)}(p) := a^{(i)} \left[p - p_{\min}^{(i)} \right]^+, \ d^{(i)}(p) := b^{(i)} \left[p_{\max}^{(i)} - p \right]^+, \tag{2.1}$$

where $x^+ := \max\{0, x\}$ $(x \in \mathbb{R})$. Peer i is entirely described by four parameters, $a^{(i)}, b^{(i)}, p_{\max}^{(i)}$, and $p_{\min}^{(i)}$. The two price parameters, $p_{\min}^{(i)}$ and $p_{\max}^{(i)}$, respectively represent the minimum value of the unit price p_o that peer i will sell some of its own disk space and the maximum value of the unit price p_s that it will pay for storage space, and $a^{(i)}$ and $b^{(i)}$ correspond to the increase in sold capacity with the unit price p_o ($\geq p_{\min}^{(i)}$) and the decrease in bought storage space with the unit price p_s ($\leq p_{\max}^{(i)}$). For a given p (≥ 0), $s^{(i)}(p)$ (resp. $d^{(i)}(p)$) is the amount of storage capacity that peer i would choose to sell (resp. buy) if peer i were paid (resp. charged) a unit price p for it.

When the supply and demand functions are defined as in (2.1), the utility function $U^{(i)}$ of peer i is of the following form (see [21, Section II] for the details),

$$\begin{aligned}
&V^{(i)}\left(c_s^{(i)}\right) := \frac{1}{b^{(i)}}\left[-\frac{\left(c_s^{(i)} \wedge b^{(i)} p_{\max}^{(i)}\right)^2}{2} + b^{(i)} p_{\max}^{(i)}\left(c_s^{(i)} \wedge b^{(i)} p_{\max}^{(i)}\right)\right],\\
&O^{(i)}\left(c_o^{(i)}\right) := \frac{1}{a^{(i)}}\frac{\left(c_o^{(i)}\right)^2}{2},\ P^{(i)}\left(c_o^{(i)}\right) := O^{(i)}\left(c_o^{(i)}\right) + p_{\min}^{(i)} c_o^{(i)},\\
&\varepsilon^{(i)} := p_s c_s^{(i)} - p_o c_o^{(i)},\\
&U^{(i)}\left(c_s^{(i)}, c_o^{(i)}, \varepsilon^{(i)}\right) := V^{(i)}\left(c_s^{(i)}\right) - P^{(i)}\left(c_o^{(i)}\right) - \varepsilon^{(i)},
\end{aligned} \tag{2.2}$$

where $x \wedge y := \min\{x, y\}$ ($x, y \in \mathbb{R}$), $V^{(i)}(c_s^{(i)})$ is peer i's valuation obtained when it uses $c_s^{(i)}$ (i.e., the price that it is willing to pay to store an amount of data $c_s^{(i)}$), $O^{(i)}(c_o^{(i)})$ is the opportunity cost of offering $c_o^{(i)}$ for other peers without using $c_o^{(i)}$ for itself, $p_{\min}^{(i)} c_o^{(i)}$ is the data transfer cost, $P^{(i)}(c_o^{(i)})$ stands for the overall non-monetary cost of peer i for offering $c_o^{(i)}$, and $\varepsilon^{(i)}$ is the monetary price paid by peer i.

On the other hand, the operator (denoted by peer 0), which manages the P2P data storage system, tries to maximize its revenue, which is the total amount that the peers are charged. Since the monetary price paid by peer i is $\varepsilon^{(i)} = p_s c_s^{(i)} - p_o c_o^{(i)}$, $c_s^{(i)} = d^{(i)}(p_s)$, and $c_o^{(i)} = s^{(i)}(p_o)$, the utility function of the operator can be represented by

$$U^{(0)}(p_s, p_o) := \sum_{i \in \mathcal{I}} \varepsilon^{(i)} = p_s \sum_{i \in \mathcal{I}} d^{(i)}(p_s) - p_o \sum_{i \in \mathcal{I}} s^{(i)}(p_o). \tag{2.3}$$

We define a performance measure, called *social welfare*, as the sum of the utility functions of all peers and the operator. From (2.2) and (2.3), social welfare can be expressed as, for all $\boldsymbol{c}_s := (c_s^{(1)}, c_s^{(2)}, \ldots, c_s^{(K)})^T, \boldsymbol{c}_o := (c_o^{(1)}, c_o^{(2)}, \ldots, c_o^{(K)})^T \in \mathbb{R}^K$,

$$\begin{aligned}
W(\boldsymbol{c}_s, \boldsymbol{c}_o) &:= \sum_{i \in \mathcal{I}} U^{(i)}\left(c_s^{(i)}, c_o^{(i)}, \varepsilon^{(i)}\right) + U^{(0)}(p_s, p_o)\\
&= \sum_{i \in \mathcal{I}} \left[V^{(i)}\left(c_s^{(i)}\right) - P^{(i)}\left(c_o^{(i)}\right)\right],
\end{aligned} \tag{2.4}$$

where $\boldsymbol{x}^T$ denotes the transpose of the vector $\boldsymbol{x}$. It is desirable to maximize W defined by (2.4) because it makes the whole system stable and reliable. We call $W^{(i)} \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined for all $(c_s^{(i)}, c_o^{(i)}) \in \mathbb{R} \times \mathbb{R}$ by

$$W^{(i)}\left(c_s^{(i)}, c_o^{(i)}\right) := V^{(i)}\left(c_s^{(i)}\right) - P^{(i)}\left(c_o^{(i)}\right)$$

the *welfare* of peer i.

3. Main Problems

Here, we describe the two incentive schemes [21, Sections II and III], a *symmetric scheme* and a *profit-oriented pricing scheme*, for controlling the P2P data storage system and point out their storage allocation problems.

3.1. Symmetric scheme.

The symmetric management scheme is based on the idea that every peer should contribute to the system in terms of service at least as much as it benefits from others. It imposes a rule that the contribution of each peer (the storage space offered by each peer) should be equal to its use of the system (the storage space it uses to store its own data). This scheme can work without an operator, and hence, does not use monetary transactions. Here, peer i tries to choose $c_s^{(i)}$ and $c_o^{(i)}$ so as to maximize its welfare $W^{(i)}$ subject to $c_o^{(i)} \geq c_s^{(i)}$ (≥ 0). Therefore, the constrained set, denoted by $C^{(i)}$ $(\subset \mathbb{R}^K \times \mathbb{R}^K)$, and the objective function, denoted by $f^{(i)} \colon \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$, of peer i $(i \in \mathcal{I})$ can be expressed as,

$$C^{(i)} := \mathbb{R}_+^K \times \mathbb{R}_+^K \cap \left\{(\boldsymbol{c}_s, \boldsymbol{c}_o) \in \mathbb{R}^K \times \mathbb{R}^K \colon c_o^{(i)} \geq c_s^{(i)}\right\}, \tag{3.1}$$

$$f^{(i)}\left(\boldsymbol{c}_s, \boldsymbol{c}_o\right) := -W^{(i)}\left(c_s^{(i)}, c_o^{(i)}\right) = -\left[V^{(i)}\left(c_s^{(i)}\right) - P^{(i)}\left(c_o^{(i)}\right)\right] \tag{3.2}$$

for all $\boldsymbol{c}_s := (c_s^{(1)}, c_s^{(2)}, \ldots, c_s^{(K)})^T, \boldsymbol{c}_o := (c_o^{(1)}, c_o^{(2)}, \ldots, c_o^{(K)})^T \in \mathbb{R}^K$, where $V^{(i)}$ and $P^{(i)}$ are defined as in (2.2) and $\mathbb{R}_+^K := \{(x^{(1)}, x^{(2)}, \ldots, x^{(K)})^T \in \mathbb{R}^K \colon x^{(i)} \geq 0 \ (i = 1, 2, \ldots, K)\}$. $f^{(i)}$ $(i \in \mathcal{I})$ defined by (3.2) satisfies the *strong convexity* condition[1] because $V^{(i)}$ and $P^{(i)}$ have quadratic forms.

Let $T^{(i)} \colon \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}^K \times \mathbb{R}^K$ $(i \in \mathcal{I})$ be a mapping defined for all $\boldsymbol{c}_s, \boldsymbol{c}_o \in \mathbb{R}^K$ by

$$T^{(i)}\left(\boldsymbol{c}_s, \boldsymbol{c}_o\right) := \frac{1}{2}\left[\left(\boldsymbol{c}_s, \boldsymbol{c}_o\right) + P_{\mathbb{R}_+^K \times \mathbb{R}_+^K}\left\{P_{\hat{C}^{(i)}}\left(\boldsymbol{c}_s, \boldsymbol{c}_o\right)\right\}\right], \tag{3.3}$$

where $\hat{C}^{(i)} := \{(\boldsymbol{c}_s, \boldsymbol{c}_o) \in \mathbb{R}^K \times \mathbb{R}^K \colon c_o^{(i)} \geq c_s^{(i)}\}$ and P_D stands for the metric projection onto a closed convex set D $(\subset \mathbb{R}^K \times \mathbb{R}^K)$.[2] $P_{\mathbb{R}_+^K \times \mathbb{R}_+^K}$ and $P_{\hat{C}^{(i)}}$ can be easily computed within a finite number of arithmetic operations because

[1] $f \colon \mathbb{R}^m \to \mathbb{R}$ is called a strongly convex function with α (α-strongly convex function) if $\alpha > 0$ exists such that, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ and for all $\lambda \in [0,1]$, $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y}) - (1/2)\alpha\lambda(1-\lambda)\|\boldsymbol{x} - \boldsymbol{y}\|^2$, where $\|\cdot\|$ stands for the norm of $\mathbb{R}^m$.

[2] The metric projection onto a closed convex set D $(\subset \mathbb{R}^m)$ is defined as follows: $P_D(\boldsymbol{x}) \in D$ and $\|\boldsymbol{x} - P_D(\boldsymbol{x})\| = \inf_{\boldsymbol{y} \in D} \|\boldsymbol{x} - \boldsymbol{y}\|$ $(\boldsymbol{x} \in \mathbb{R}^m)$. P_D satisfies the nonexpansivity condition [2, Proposition 2.10]; i.e., $\|P_D(\boldsymbol{x}) - P_D(\boldsymbol{y})\| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$.

$\mathbb{R}_+^K \times \mathbb{R}_+^K$ and $\hat{C}^{(i)}$ are half-spaces of $\mathbb{R}^K \times \mathbb{R}^K$ [2, p. 406], [3, Subchapter 28.3]. $T^{(i)}$ $(i \in \mathcal{I})$ defined by (3.3) satisfies the *firm nonexpansivity* condition[3] because $P_{\mathbb{R}_+^K \times \mathbb{R}_+^K}$ and $P_{\hat{C}^{(i)}}$ are nonexpansive.[4] Moreover, $C^{(i)}$ $(i \in \mathcal{I})$ defined by (3.1) can be represented as the *fixed point set* of $T^{(i)}$ defined by (3.3); i.e.,

$$\mathrm{Fix}\left(T^{(i)}\right) := \left\{(\boldsymbol{c}_s, \boldsymbol{c}_o) \in \mathbb{R}^K \times \mathbb{R}^K : T^{(i)}\left(\boldsymbol{c}_s, \boldsymbol{c}_o\right) = (\boldsymbol{c}_s, \boldsymbol{c}_o)\right\} = C^{(i)}.$$

This is because $\mathrm{Fix}(T^{(i)}) = \mathrm{Fix}(P_{\mathbb{R}_+^K \times \mathbb{R}_+^K} P_{\hat{C}^{(i)}}) = \mathbb{R}_+^K \times \mathbb{R}_+^K \cap \hat{C}^{(i)} =: C^{(i)}$.

The constrained set and objective function of the operator (peer 0) can be expressed as,

$$C^{(0)} := \mathbb{R}^K \times \mathbb{R}^K = \mathrm{Fix}\left(\mathrm{Id}\right) =: \mathrm{Fix}\left(T^{(0)}\right),\ f^{(0)}(\boldsymbol{c}_s, \boldsymbol{c}_o) := 0$$

for all $(\boldsymbol{c}_s, \boldsymbol{c}_o) \in \mathbb{R}^K \times \mathbb{R}^K$, because the operator does not directly control the system. This means that control algorithms of the symmetric scheme must be implemented without the operator so as to maximize the social welfare.

Therefore, we can describe the storage allocation problem of the symmetric scheme as follows.

Problem 3.1 (storage allocation problem under symmetric scheme)**.**

$$\textit{Maximize } W\left(\boldsymbol{c}_s, \boldsymbol{c}_o\right) = -\sum_{i \in \mathcal{I}} f^{(i)}\left(\boldsymbol{c}_s, \boldsymbol{c}_o\right)$$

$$\textit{subject to } \left(\boldsymbol{c}_s, \boldsymbol{c}_o\right) \in \bigcap_{i \in \mathcal{I}} \left\{(\boldsymbol{c}_s, \boldsymbol{c}_o) \in \mathbb{R}_+^K \times \mathbb{R}_+^K : c_o^{(i)} \geq c_s^{(i)}\right\} = \bigcap_{i \in \mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right),$$

where $f^{(i)} \colon \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ *and* $T^{(i)} \colon \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}^K \times \mathbb{R}^K$ $(i \in \mathcal{I})$ *are defined as in* (3.2) *and* (3.3)*, respectively.*

Problem 3.1 is one of maximizing the social welfare W defined in (2.4) under the condition that each peer offers a storage capacity larger than the capacity used for storing its own data.

Under the symmetric scheme, each peer can communicate with a neighbor peer via the network. Hence, Problem 3.1 can be solved by *incremental optimization algorithms* (see, e.g., [7, Subchapter 8.2], [8, 18, 19, 22]) that allow each peer to use only its own private information[5] and the transmitted information from the neighbor peer. Moreover, peer i $(i \in \mathcal{I})$ tries to minimize only $f^{(i)}$ (i.e., maximize only its own welfare $W^{(i)}$) over its own

---

[3] $T \colon \mathbb{R}^m \to \mathbb{R}^m$ is called a firmly nonexpansive mapping if, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $\|T(\boldsymbol{x}) - T(\boldsymbol{y})\|^2 \leq \langle \boldsymbol{x} - \boldsymbol{y}, T(\boldsymbol{x}) - T(\boldsymbol{y})\rangle$, where $\langle \cdot, \cdot \rangle$ stands for the inner product of $\mathbb{R}^m$. $\mathrm{Fix}(T) := \{\boldsymbol{x} \in \mathbb{R}^m \colon T(\boldsymbol{x}) = \boldsymbol{x}\}$ is closed and convex when T is nonexpansive [15, Proposition 5.3].

[4] $T := (1/2)(\mathrm{Id} + S)$ satisfies the firm nonexpansivity condition when S is nonexpansive [3, Definition 4.1, Proposition 4.2], where Id stands for the identity mapping.

[5] Peer i in Problem 3.1 has its own private $f^{(i)}$ defined by (3.2) because the four parameters, $a^{(i)}, b^{(i)}, p_{\max}^{(i)}$, and $p_{\min}^{(i)}$, are its own private information.

constraint $\mathrm{Fix}(T^{(i)}) = C^{(i)}$. Accordingly, each peer never uses information including other peers' objective functions and constrained sets. Therefore, none of the peers can use the metric projection P_C onto the polytope $C := \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$. In this paper, we present an algorithm for solving Problem 3.1 that is different from the conventional incremental optimization algorithms which use P_C. We will show that the algorithm converges to the solution to Problem 3.1 under certain assumptions (Section 4).

3.2. **Profit-oriented pricing scheme.** A payment-based management scheme is based on monetary exchanges where peers can buy storage space in the system for a unit price p_s and sell some of their disk capacity for a unit price p_o. Assuming that the operator knows that peer i ($i \in \mathcal{I}$) will sell $s^{(i)}(p_o)$ and buy $d^{(i)}(p_s)$, it tries to choose p_s and p_o so as to maximize its profit $U^{(0)}(p_s, p_o)$. Accordingly, the constrained set and objective function of the operator (peer 0) are defined as follows.

$$C^{(0)} := \mathbb{R}_+ \times \mathbb{R}_+ \cap \left\{ (p_s, p_o) \in \mathbb{R}\times\mathbb{R} \colon \sum_{i\in\mathcal{I}} s^{(i)}(p_o) \geq \sum_{i\in\mathcal{I}} d^{(i)}(p_s) \right\}, \tag{3.4}$$

$$f^{(0)}(p_s, p_o) := -U^{(0)}(p_s, p_o) = -\left[p_s \sum_{i\in\mathcal{I}} d^{(i)}(p_s) - p_o \sum_{i\in\mathcal{I}} s^{(i)}(p_o) \right] \tag{3.5}$$

for all $(p_s, p_o) \in \mathbb{R}\times\mathbb{R}$.

$C^{(0)}$ defined in (3.4) is an absolute set in which conditions are needed to control the system. This is because $\sum_{i\in\mathcal{I}} c_s^{(i)} = \sum_{i\in\mathcal{I}} d^{(i)}(p_s)$, which is used for storing data, must not exceed the sum offered by peers, i.e., $\sum_{i\in\mathcal{I}} c_o^{(i)} = \sum_{i\in\mathcal{I}} s^{(i)}(p_o)$.

Here, let us define a mapping $T^{(0)} \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ for all $(p_s, p_o) \in \mathbb{R}\times\mathbb{R}$ by

$$T^{(0)}(p_s, p_o) := \frac{1}{2}\left[(p_s, p_o) + P_{\mathbb{R}_+\times\mathbb{R}_+}\left\{ P_{\hat{C}^{(0)}}(p_s, p_o) \right\} \right], \tag{3.6}$$

where $\hat{C}^{(0)} := \{(p_s, p_o) \in \mathbb{R}\times\mathbb{R} \colon \sum_{i\in\mathcal{I}} s^{(i)}(p_o) \geq \sum_{i\in\mathcal{I}} d^{(i)}(p_s)\}$. Since $s^{(i)}$ and $d^{(i)}$ defined as in (2.1) are affine, $\hat{C}^{(0)}$ is a half-space, which means that $P_{\hat{C}^{(0)}}$ can be easily computed within a finite number of arithmetic operations. $T^{(0)}$ defined in (3.6) satisfies the firm nonexpansivity condition (see Footnotes 3 and 4), and

$$\mathrm{Fix}\left(T^{(0)}\right) := \left\{ (p_s, p_o) \in \mathbb{R}\times\mathbb{R} \colon T^{(0)}(p_s, p_o) = (p_s, p_o) \right\} = C^{(0)}$$

because $\mathrm{Fix}(T^{(0)}) = \mathrm{Fix}(P_{\mathbb{R}_+\times\mathbb{R}_+} P_{\hat{C}^{(0)}}) = \mathbb{R}_+ \times \mathbb{R}_+ \cap \hat{C}^{(0)} =: C^{(0)}$ (see also the discussion in (3.3)). Moreover, since $s^{(i)}$ and $d^{(i)}$ in (2.1) are affine, $f^{(0)}$ in (3.5) satisfies the strong convexity condition.

Meanwhile, peer i ($i \in \mathcal{I}$) selfishly chooses strategies that maximize its welfare $W^{(i)}$. Accordingly, the constrained set and objective function of peer

i $(i \in \mathcal{I})$ can be expressed as

$$C^{(i)} := \left[p_{\min}^{(i)}, p_{\max}^{(i)}\right] \times \left[p_{\min}^{(i)}, p_{\max}^{(i)}\right] = \mathrm{Fix}\left(P_{C^{(i)}}\right) =: \mathrm{Fix}\left(T^{(i)}\right), \tag{3.7}$$

$$f^{(i)}\left(p_s, p_o\right) := -\left[V^{(i)}\left(d^{(i)}\left(p_s\right)\right) - P^{(i)}\left(s^{(i)}\left(p_o\right)\right)\right] \tag{3.8}$$

for all $(p_s, p_o) \in \mathbb{R} \times \mathbb{R}$. Since $s^{(i)}$ and $d^{(i)}$ in (2.1) are affine, and $V^{(i)}$ and $P^{(i)}$ have quadratic forms, $f^{(i)}$ $(i \in \mathcal{I})$ in (3.8) satisfies the strong convexity condition. $T^{(i)} := P_{C^{(i)}}$ $(i \in \mathcal{I})$ in (3.7) is firmly nonexpansive [2, Facts 1.5].

The main objective of the profit-oriented pricing scheme is to determine optimal prices p_s and p_o so as to maximize the operator's profit $U^{(0)}$. Meanwhile, it is desirable to maximize the social welfare W to make the whole system stable and reliable. As such, we can pose the storage allocation problem under the profit-oriented pricing scheme as one of maximizing the weighted mean of the operator's profit and social welfare, $\lambda U^{(0)} + (1-\lambda)W$, for some weight parameter λ $(\in (0,1))$.

Problem 3.2 (storage allocation problem under profit-oriented pricing scheme)**.**

$$\textit{Maximize } \lambda U^{(0)}\left(p_s, p_o\right) + (1-\lambda) W\left(p_s, p_o\right) = -\left[\lambda f^{(0)} + (1-\lambda)\sum_{i \in \mathcal{I}} f^{(i)}\right]\left(p_s, p_o\right)$$

$$\textit{subject to } \left(p_s, p_o\right) \in \left\{\left(p_s, p_o\right) \in \mathbb{R}_+ \times \mathbb{R}_+ \colon \sum_{i \in \mathcal{I}} s^{(i)}\left(p_o\right) \geq \sum_{i \in \mathcal{I}} d^{(i)}\left(p_s\right)\right\}$$
$$\cap \bigcap_{i \in \mathcal{I}} \left[p_{\min}^{(i)}, p_{\max}^{(i)}\right] \times \left[p_{\min}^{(i)}, p_{\max}^{(i)}\right] = \bigcap_{i \in \{0\} \cup \mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right),$$

where $\lambda \in (0,1)$ *is a parameter chosen in advance, and* $f^{(i)} \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and* $T^{(i)} \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ $(i \in \{0\} \cup \mathcal{I})$ *are defined as in* (3.5)*,* (3.6)*,* (3.7)*, and* (3.8)*.*

Problem 3.2 can be solved if one assumes the operator can communicate with all peers and has access to a point computed by $f^{(i)}$ and $T^{(i)}$ of peer i $(i \in \mathcal{I})$. This implies that the operator can use *broadcast optimization algorithms* (see, e.g., [11, 12]). In this paper, we will present a broadcast optimization algorithm for solving Problem 3.2 that is different from the conventional broadcast optimization algorithms [11, 12] which use the proximity operator of $f^{(i)}$, and show that the proposed algorithm converges to the solution to Problem 3.2 under certain assumptions (Section 5).

The following propositions will be used to prove the main theorems.

Proposition 3.1. [31, Lemma 3.1] *Suppose that* $f \colon \mathbb{R}^m \to \mathbb{R}$ *is* α*-strongly convex and differentiable,* $\nabla f \colon \mathbb{R}^m \to \mathbb{R}^m$ *is* L*-Lipschitz continuous,*[6] $\mu \in$

[6] $A \colon \mathbb{R}^m \to \mathbb{R}^m$ is said to be L-Lipschitz continuous if $\|A(\boldsymbol{x}) - A(\boldsymbol{y})\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|$ $(\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m)$.

$(0, 2\alpha/L^2)$, *and* $S := \mathrm{Id} - \mu\lambda\nabla f$, *where* $\lambda \in [0,1]$. *Then, for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $\|S(\boldsymbol{x}) - S(\boldsymbol{y})\| \leq (1-\tau\lambda)\|\boldsymbol{x}-\boldsymbol{y}\|$, *where* $\tau := 1 - \sqrt{1-\mu(2\alpha - \mu L^2)} \in (0,1]$.

Proposition 3.2. [4, Lemma 1.2] *Assume that* $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ *satisfies* $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\beta_n$ $(n \in \mathbb{N})$, *where* $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1]$ *and* $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *with* $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\limsup_{n\to\infty}\beta_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

Suppose that $T\colon \mathbb{R}^m \to \mathbb{R}^m$ is firmly nonexpansive; i.e., $\|T(\boldsymbol{x}) - T(\boldsymbol{y})\|^2 \leq \langle \boldsymbol{x}-\boldsymbol{y}, T(\boldsymbol{x}) - T(\boldsymbol{y})\rangle$ $(\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^m)$. From $\langle \boldsymbol{x},\boldsymbol{y}\rangle = (1/2)\{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2\}$, we have $\|T(\boldsymbol{x}) - T(\boldsymbol{y})\|^2 \leq \langle \boldsymbol{x}-\boldsymbol{y}, T(\boldsymbol{x}) - T(\boldsymbol{y})\rangle = (1/2)\{\|\boldsymbol{x}-\boldsymbol{y}\|^2 + \|T(\boldsymbol{x}) - T(\boldsymbol{y})\|^2 - \|(\boldsymbol{x}-\boldsymbol{y}) - (T(\boldsymbol{x}) - T(\boldsymbol{y}))\|^2\}$ $(\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^m)$. This leads us to the following.

Proposition 3.3. *Suppose that* $T\colon \mathbb{R}^m \to \mathbb{R}^m$ *is firmly nonexpansive. Then, for all* $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^m$, $\|T(\boldsymbol{x}) - T(\boldsymbol{y})\|^2 \leq \|\boldsymbol{x}-\boldsymbol{y}\|^2 - \|(\boldsymbol{x}-\boldsymbol{y}) - (T(\boldsymbol{x}) - T(\boldsymbol{y}))\|^2$.

4. Distributed Control under the Symmetric Scheme

This section considers the following problem.

$$\text{Minimize } \sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}) \text{ subject to } \boldsymbol{c} \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right), \tag{4.1}$$

where $f^{(i)}\colon \mathbb{R}^m \to \mathbb{R}$ $(i \in \mathcal{I} := \{1,2,\ldots,K\})$ is $\alpha^{(i)}$-strongly convex and differentiable, $\nabla f^{(i)}\colon \mathbb{R}^m \to \mathbb{R}^m$ is $L^{(i)}$-Lipschitz continuous, and $T^{(i)}\colon \mathbb{R}^m \to \mathbb{R}^m$ $(i \in \mathcal{I})$ is firmly nonexpansive with $\bigcap_{i\in\mathcal{I}}\mathrm{Fix}(T^{(i)}) \neq \emptyset$. Subsection 3.1 tells us that Problem 3.1 coincides with problem (4.1) when $m := 2K$ and $f^{(i)}$ and $T^{(i)}$ are defined by (3.2) and (3.3). Moreover, since $\sum_{i\in\mathcal{I}} f^{(i)}$ is strongly convex and Lipschitz continuous, and $\bigcap_{i\in\mathcal{I}}\mathrm{Fix}(T^{(i)})$ is closed and convex, problem (4.1) has a unique solution [31, Proposition 2.7].

Here, we assume the following.

Assumption 4.1. *Peer* i $(i \in \mathcal{I})$ *uses* $\mu \in (0, \min_{i\in\mathcal{I}} 2\alpha^{(i)}/L^{(i)^2})$ *and the sequence* $(\lambda_n)_{n\in\mathbb{N}} \subset (0,1)$ *satisfying*[7]

$$\text{(C1)} \lim_{n\to\infty}\lambda_n = 0,\ \text{(C2)} \sum_{n=0}^{\infty}\lambda_n = \infty,\ \text{(C3)} \lim_{n\to\infty}\frac{1}{\lambda_{n+1}}\left|\frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_n}\right| = 0.$$

The following algorithm can be used to solve problem (4.1).

Algorithm 4.1 (incremental gradient algorithm)**.**

Step 0. Peer K *sets* $\boldsymbol{c}_0 \in \mathbb{R}^m$ *arbitrarily and transmits* $\boldsymbol{c}_0^{(0)} := \boldsymbol{c}_0$ *to peer* 1.

Step 1. Given $\boldsymbol{c}_n := \boldsymbol{c}_n^{(0)} \in \mathbb{R}^m$, *peer* i *computes* $\boldsymbol{c}_n^{(i)} \in \mathbb{R}^m$ *cyclically by*

$$\boldsymbol{c}_n^{(i)} := T^{(i)}\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right)\ (i = 1,2,\ldots,K).$$

Step 2. Peer K *sets* $\boldsymbol{c}_{n+1} \in \mathbb{R}^m$ *by* $\boldsymbol{c}_{n+1} := \boldsymbol{c}_n^{(K)}$ *and transmits* $\boldsymbol{c}_{n+1}^{(0)} := \boldsymbol{c}_{n+1}$ *to peer* 1. *Put* $n := n+1$, *and go to Step 1.*

[7] Example of $(\lambda_n)_{n\in\mathbb{N}}$ satisfying (C1)–(C3) is $\lambda_n := 1/(n+1)^a$ $(a \in (0,1/2))$.

We are in the position to perform the convergence analysis on Algorithm 4.1.

Theorem 4.1. *Under Assumption 4.1, the sequence* $(\boldsymbol{c}_n^{(i)})_{n\in\mathbb{N}}$ $(i \in \mathcal{I})$ *generated by Algorithm 4.1 converges to the solution to problem* (4.1).

Theorem 4.1 means that each peer that uses Algorithm 4.1 with $f^{(i)}(\boldsymbol{c}) := f^{(i)}(\boldsymbol{c}_s, \boldsymbol{c}_o)$ and $T^{(i)}(\boldsymbol{c}) := T^{(i)}(\boldsymbol{c}_s, \boldsymbol{c}_o)$ defined by (3.2) and (3.3) can solve the storage allocation problem 3.1 under the symmetric scheme. It would be difficult for all peers to set $\mu \in (0, \min_{i\in\mathcal{I}} 2\alpha^{(i)}/{L^{(i)}}^2)$ in advance because μ depends on all $\alpha^{(i)}$s and $L^{(i)}$s. Even if $\mu \geq \min_{i\in\mathcal{I}} 2\alpha^{(i)}/{L^{(i)}}^2$, (C1) guarantees that $n_0 \in \mathbb{N}$ exists such that $\mu\lambda_n < \min_{i\in\mathcal{I}} 2\alpha^{(i)}/{L^{(i)}}^2$ for all $n \geq n_0$. Hence, Theorem 4.1 ensures that $(\boldsymbol{c}_n^{(i)})_{n\geq n_0}$ $(i \in \mathcal{I})$ in Algorithm 4.1 converges to the unique solution to problem (4.1). This implies that Algorithm 4.1 can solve problem (4.1) without depending on the choice of μ. See Section 6 for the behaviors of Algorithm 4.1 with different values of μ.

Proof. We first show that $(\boldsymbol{c}_n^{(i)})_{n\in\mathbb{N}}$ and $(\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)}))_{n\in\mathbb{N}}$ $(i \in \mathcal{I})$ are bounded. Choose $\boldsymbol{c} \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$ arbitrarily, and put $\tau^{(i)} := 1 - \sqrt{1 - \mu(2\alpha^{(i)} - \mu {L^{(i)}}^2)}$, $\tau := \min_{i\in\mathcal{I}} \tau^{(i)}$, and $M_1 := \max_{i\in\mathcal{I}} \|\nabla f^{(i)}(\boldsymbol{c})\|$. The nonexpansivity of $T^{(i)}$ guarantees that, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\left\|\boldsymbol{c}_n^{(i)} - \boldsymbol{c}\right\| &= \left\|T^{(i)}\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right) - T^{(i)}(\boldsymbol{c})\right\| \\
&\leq \left\|\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right) - \boldsymbol{c}\right\| \\
&= \left\|\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right) - \left(\boldsymbol{c} - \mu\lambda_n \nabla f^{(i)}(\boldsymbol{c})\right) - \mu\lambda_n \nabla f^{(i)}(\boldsymbol{c})\right\| \\
&\leq \left\|\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right) - \left(\boldsymbol{c} - \mu\lambda_n \nabla f^{(i)}(\boldsymbol{c})\right)\right\| + \mu M_1 \lambda_n,
\end{aligned}
$$

which from $\mu < 2L^{(i)}/{\alpha^{(i)}}^2$, $\tau \leq \tau^{(i)}$, and Proposition 3.1 implies that, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\left\|\boldsymbol{c}_n^{(i)} - \boldsymbol{c}\right\| &\leq \left(1 - \tau^{(i)}\lambda_n\right)\left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right\| + \mu M_1 \lambda_n \\
&\leq (1 - \tau\lambda_n)\left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right\| + \mu M_1 \lambda_n.
\end{aligned}
\tag{4.2}
$$

Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|\boldsymbol{c}_{n+1} - \boldsymbol{c}\| &= \left\|\boldsymbol{c}_n^{(K)} - \boldsymbol{c}\right\| \\
&\leq (1-\tau\lambda_n)\left\|\boldsymbol{c}_n^{(K-1)} - \boldsymbol{c}\right\| + \mu M_1 \lambda_n \\
&\leq (1-\tau\lambda_n)\left\{(1-\tau\lambda_n)\left\|\boldsymbol{c}_n^{(K-2)} - \boldsymbol{c}\right\| + \mu M_1 \lambda_n\right\} + \mu M_1 \lambda_n \\
&\leq (1-\tau\lambda_n)^2\left\|\boldsymbol{c}_n^{(K-2)} - \boldsymbol{c}\right\| + 2\mu M_1 \lambda_n \\
&\leq (1-\tau\lambda_n)^K\left\|\boldsymbol{c}_n^{(0)} - \boldsymbol{c}\right\| + K\mu M_1 \lambda_n \\
&\leq (1-\tau\lambda_n)\|\boldsymbol{c}_n - \boldsymbol{c}\| + \left(\frac{K\mu M_1}{\tau}\right)\tau\lambda_n.
\end{aligned}$$

Induction shows that, for all $n \in \mathbb{N}$,

$$\|\boldsymbol{c}_n - \boldsymbol{c}\| \leq \max\left\{\|\boldsymbol{c}_0 - \boldsymbol{c}\|, \frac{K\mu M_1}{\tau}\right\}.$$

This means $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ $(= (\boldsymbol{c}_n^{(0)})_{n\in\mathbb{N}})$ is bounded. Hence, from (4.2) when $i = 1$, $(\boldsymbol{c}_n^{(1)})$ is also bounded. Accordingly, induction shows that $(\boldsymbol{c}_n^{(i)})$ $(i \in \mathcal{I})$ is bounded. Moreover, from $\|\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)}) - \nabla f^{(i)}(\boldsymbol{c})\| \leq L^{(i)}\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\|$ $(i \in \mathcal{I}, n \in \mathbb{N})$ and the boundedness of $(\boldsymbol{c}_n^{(i)})$ $(i \in \mathcal{I})$, $(\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)}))_{n\in\mathbb{N}}$ $(i \in \mathcal{I})$ is bounded.

The nonexpansivity of $T^{(i)}$ guarantees that, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left\|\boldsymbol{c}_{n+1}^{(i)} - \boldsymbol{c}_n^{(i)}\right\| \\
&= \left\|T^{(i)}\left(\boldsymbol{c}_{n+1}^{(i-1)} - \mu\lambda_{n+1}\nabla f^{(i)}\left(\boldsymbol{c}_{n+1}^{(i-1)}\right)\right) - T^{(i)}\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right)\right\| \\
&\leq \left\|\left(\boldsymbol{c}_{n+1}^{(i-1)} - \mu\lambda_{n+1}\nabla f^{(i)}\left(\boldsymbol{c}_{n+1}^{(i-1)}\right)\right) - \left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right)\right\| \\
&\leq \left\|\left(\boldsymbol{c}_{n+1}^{(i-1)} - \mu\lambda_{n+1}\nabla f^{(i)}\left(\boldsymbol{c}_{n+1}^{(i-1)}\right)\right) - \left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_{n+1}\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right)\right\| \\
&\quad + \mu|\lambda_n - \lambda_{n+1}|\left\|\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\|,
\end{aligned}$$

which from Proposition 3.1 means that, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\left\|\boldsymbol{c}_{n+1}^{(i)} - \boldsymbol{c}_n^{(i)}\right\| \leq (1-\tau\lambda_{n+1})\left\|\boldsymbol{c}_{n+1}^{(i-1)} - \boldsymbol{c}_n^{(i-1)}\right\| + M_2|\lambda_n - \lambda_{n+1}|,$$

where $M_2 := \max_{i\in\mathcal{I}}(\sup_{n\in\mathbb{N}} \mu\|\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)})\|) < \infty$. Therefore, we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\boldsymbol{c}_{n+1} - \boldsymbol{c}_n\| &= \left\|\boldsymbol{c}_n^{(K)} - \boldsymbol{c}_{n-1}^{(K)}\right\| \\ &\le (1-\tau\lambda_n)\left\|\boldsymbol{c}_n^{(K-1)} - \boldsymbol{c}_{n-1}^{(K-1)}\right\| + M_2\left|\lambda_n - \lambda_{n-1}\right| \\ &\le (1-\tau\lambda_n)^K\left\|\boldsymbol{c}_n^{(0)} - \boldsymbol{c}_{n-1}^{(0)}\right\| + KM_2\left|\lambda_n - \lambda_{n-1}\right| \\ &\le (1-\tau\lambda_n)\left\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\right\| + KM_2\left|\lambda_n - \lambda_{n-1}\right|,\end{aligned}$$

which from $M_3 := \sup_{n\in\mathbb{N}}\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\| < \infty$ and $1 \le 1/\lambda_{n-1}$ implies

$$\begin{aligned}\frac{\|\boldsymbol{c}_{n+1} - \boldsymbol{c}_n\|}{\lambda_n} &\le (1-\tau\lambda_n)\frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_n} + KM_2\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} \\ &= (1-\tau\lambda_n)\frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_{n-1}} + (1-\tau\lambda_n)\left\{\frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_n} - \frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_{n-1}}\right\} \\ &\quad + KM_2\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} \\ &\le (1-\tau\lambda_n)\frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_{n-1}} + M_3\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| + KM_2\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} \\ &\le (1-\tau\lambda_n)\frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_{n-1}} + M_3\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| + KM_2\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n\lambda_{n-1}} \\ &= (1-\tau\lambda_n)\frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_{n-1}} + (M_3 + KM_2)\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| \\ &= (1-\tau\lambda_n)\frac{\|\boldsymbol{c}_n - \boldsymbol{c}_{n-1}\|}{\lambda_{n-1}} + \frac{M_3 + KM_2}{\tau}\tau\lambda_n\frac{1}{\lambda_n}\left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right|.\end{aligned}$$

Hence, from (C2), (C3), and Proposition 3.2, we have

$$\lim_{n\to\infty}\frac{\|\boldsymbol{c}_{n+1} - \boldsymbol{c}_n\|}{\lambda_n} = 0. \tag{4.3}$$

Accordingly, (C1) guarantees that

$$\lim_{n\to\infty}\|\boldsymbol{c}_{n+1} - \boldsymbol{c}_n\| = 0. \tag{4.4}$$

Choose $\boldsymbol{c} \in \bigcap_{i\in\mathcal{I}}\mathrm{Fix}(T^{(i)})$ arbitrarily; i.e., $\boldsymbol{c} = T^{(i)}(\boldsymbol{c})$ $(i \in \mathcal{I})$. From $\boldsymbol{c}_n^{(i)} := T^{(i)}(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)}))$ and the firm nonexpansivity of $T^{(i)}$, Proposition 3.3 ensures that, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}&\left\|\boldsymbol{c}_n^{(i)} - \boldsymbol{c}\right\|^2 \\ &\le \left\|\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right) - \boldsymbol{c}\right\|^2 \\ &\quad - \left\|\left(\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right) - \boldsymbol{c}\right) - \left(\boldsymbol{c}_n^{(i)} - \boldsymbol{c}\right)\right\|^2 \\ &= \left\|\left(\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right) - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\|^2 - \left\|\left(\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\right) - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\|^2,\end{aligned}$$

which from $\|\boldsymbol{x}-\boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 - 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle + \|\boldsymbol{y}\|^2$ ($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$) implies

$$\begin{aligned}\left\|\boldsymbol{c}_n^{(i)} - \boldsymbol{c}\right\|^2 \leq \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right\|^2 - 2\mu\lambda_n \left\langle \boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}, \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\rangle - \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\right\|^2 \\ + 2\mu\lambda_n \left\langle \boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}, \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\rangle \\ \leq \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right\|^2 - \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\right\|^2 + M_4\lambda_n,\end{aligned}$$

where $M_4 := \max_{i\in\mathcal{I}}(\sup_{n\in\mathbb{N}} 2\mu|\langle \boldsymbol{c} - \boldsymbol{c}_n^{(i)}, \nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)})\rangle|) < \infty$. Accordingly, we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|\boldsymbol{c}_{n+1} - \boldsymbol{c}\|^2 &= \left\|\boldsymbol{c}_n^{(K)} - \boldsymbol{c}\right\|^2 \\ &\leq \left\|\boldsymbol{c}_n^{(K-1)} - \boldsymbol{c}\right\|^2 - \left\|\boldsymbol{c}_n^{(K-1)} - \boldsymbol{c}_n^{(K)}\right\|^2 + M_4\lambda_n \\ &\leq \left\|\boldsymbol{c}_n^{(0)} - \boldsymbol{c}\right\|^2 - \sum_{i\in\mathcal{I}} \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\right\|^2 + KM_4\lambda_n \\ &= \|\boldsymbol{c}_n - \boldsymbol{c}\|^2 - \sum_{i\in\mathcal{I}} \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\right\|^2 + KM_4\lambda_n,\end{aligned}$$

which means

$$\begin{aligned}\sum_{i\in\mathcal{I}} \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\right\|^2 &\leq \|\boldsymbol{c}_n - \boldsymbol{c}\|^2 - \|\boldsymbol{c}_{n+1} - \boldsymbol{c}\|^2 + KM_4\lambda_n \\ &= (\|\boldsymbol{c}_n - \boldsymbol{c}\| + \|\boldsymbol{c}_{n+1} - \boldsymbol{c}\|)(\|\boldsymbol{c}_n - \boldsymbol{c}\| - \|\boldsymbol{c}_{n+1} - \boldsymbol{c}\|) + KM_4\lambda_n \\ &\leq M_5 \|\boldsymbol{c}_n - \boldsymbol{c}_{n+1}\| + KM_4\lambda_n,\end{aligned}$$

where $M_5 := \sup_{n\in\mathbb{N}}(\|\boldsymbol{c}_n - \boldsymbol{c}\| + \|\boldsymbol{c}_{n+1} - \boldsymbol{c}\|) < \infty$. Equation (4.4) and (C1) lead us to that $\lim_{n\to\infty}\sum_{i\in\mathcal{I}} \|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\|^2 = 0$; i.e., $\lim_{n\to\infty}\|\boldsymbol{c}_n^{(i)} - \boldsymbol{c}_n^{(i-1)}\| = 0$ ($i \in \mathcal{I}$). Since $\|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i-1)}\| = \|\boldsymbol{c}_n^{(0)} - \boldsymbol{c}_n^{(i-1)}\| \leq \sum_{j=1}^{i-1}\|\boldsymbol{c}_n^{(j-1)} - \boldsymbol{c}_n^{(j)}\|$ ($i \in \mathcal{I}, n \in \mathbb{N}$), we have

$$\lim_{n\to\infty}\left\|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i-1)}\right\| = 0 \ \ (i \in \mathcal{I}). \tag{4.5}$$

Moreover, since $\|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i)}\| \leq \|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i-1)}\| + \|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n^{(i)}\|$ ($i \in \mathcal{I}, n \in \mathbb{N}$), we also find that $\lim_{n\to\infty}\|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i)}\| = 0$ ($i \in \mathcal{I}$). The nonexpansivity of $T^{(i)}$ ensures that, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$, $\|\boldsymbol{c}_n^{(i)} - T^{(i)}(\boldsymbol{c}_n)\| = \|T^{(i)}(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)})) - T^{(i)}(\boldsymbol{c}_n)\| \leq \|(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)})) - \boldsymbol{c}_n\| \leq \|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}_n\| + \mu\lambda_n\|\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)})\|$. Equation (4.5), the boundedness of $(\nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)}))_{n\in\mathbb{N}}$, and (C1) guarantee that $\lim_{n\to\infty}\|\boldsymbol{c}_n^{(i)} - T^{(i)}(\boldsymbol{c}_n)\| = 0$ ($i \in \mathcal{I}$). From $\|\boldsymbol{c}_n - T^{(i)}(\boldsymbol{c}_n)\| \leq \|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i)}\| + \|\boldsymbol{c}_n^{(i)} - T^{(i)}(\boldsymbol{c}_n)\|$ ($i \in \mathcal{I}, n \in \mathbb{N}$), and $\lim_{n\to\infty}\|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i)}\| = \lim_{n\to\infty}\|\boldsymbol{c}_n^{(i)} - T^{(i)}(\boldsymbol{c}_n)\| = 0$ ($i \in \mathcal{I}$), we get

$$\lim_{n\to\infty}\left\|\boldsymbol{c}_n - T^{(i)}(\boldsymbol{c}_n)\right\| = 0 \ \ (i \in \mathcal{I}). \tag{4.6}$$

The boundedness of $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ guarantees the existence of an accumulation point of $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$. Let $\boldsymbol{c}^* \in \mathbb{R}^m$ be an arbitrary accumulation point of $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$. Accordingly, a subsequence $(\boldsymbol{c}_{n_k})_{k\in\mathbb{N}}$ of $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ exists such that $(\boldsymbol{c}_{n_k})_{k\in\mathbb{N}}$ converges to $\boldsymbol{c}^*$. Hence, the continuity of $T^{(i)}$ and (4.6) imply that

$$0 = \lim_{k\to\infty} \left\| \boldsymbol{c}_{n_k} - T^{(i)}\left(\boldsymbol{c}_{n_k}\right)\right\| = \left\|\boldsymbol{c}^* - T^{(i)}\left(\boldsymbol{c}^*\right)\right\| \quad (i \in \mathcal{I});$$

i.e., $\boldsymbol{c}^* \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$.

The nonexpansivity of $T^{(i)}$ guarantees that, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left\|\boldsymbol{c}_n^{(i)} - \boldsymbol{c}\right\|^2 \\
&= \left\|T^{(i)}\left(\boldsymbol{c}_n^{(i-1)} - \mu\lambda_n \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right) - T^{(i)}\left(\boldsymbol{c}\right)\right\|^2 \\
&\leq \left\|\left(\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right) - \mu\lambda_n \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\|^2 \\
&= \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right\|^2 + 2\mu\lambda_n \left\langle \boldsymbol{c} - \boldsymbol{c}_n^{(i-1)}, \nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\rangle + \mu^2\lambda_n^2 \left\|\nabla f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right\|^2.
\end{aligned}$$

Since the gradient of $f^{(i)}$ $(i \in \mathcal{I})$ at $\boldsymbol{x} \in \mathbb{R}^m$ satisfies $f^{(i)}(\boldsymbol{y}) \geq f^{(i)}(\boldsymbol{x}) + \langle \boldsymbol{y} - \boldsymbol{x}, \nabla f^{(i)}(\boldsymbol{x})\rangle$ $(\boldsymbol{y} \in \mathbb{R}^m)$, we have $\langle \boldsymbol{c} - \boldsymbol{c}_n^{(i-1)}, \nabla f^{(i)}(\boldsymbol{c}_n^{(i-1)})\rangle \leq f^{(i)}(\boldsymbol{c}) - f^{(i)}(\boldsymbol{c}_n^{(i-1)})$ $(i \in \mathcal{I}, n \in \mathbb{N})$. Thus, for all $i \in \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\left\|\boldsymbol{c}_n^{(i)} - \boldsymbol{c}\right\|^2 \leq \left\|\boldsymbol{c}_n^{(i-1)} - \boldsymbol{c}\right\|^2 + 2\mu\lambda_n\left[f^{(i)}\left(\boldsymbol{c}\right) - f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right] + M_2^2\lambda_n^2.$$

Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\|\boldsymbol{c}_{n+1} - \boldsymbol{c}\right\|^2 &\leq \left\|\boldsymbol{c}_n^{(K-1)} - \boldsymbol{c}\right\|^2 + 2\mu\lambda_n\left[f^{(K)}\left(\boldsymbol{c}\right) - f^{(K)}\left(\boldsymbol{c}_n^{(K-1)}\right)\right] + M_2^2\lambda_n^2 \\
&\leq \left\|\boldsymbol{c}_n - \boldsymbol{c}\right\|^2 + 2\mu\lambda_n \sum_{i\in\mathcal{I}}\left[f^{(i)}\left(\boldsymbol{c}\right) - f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right] + KM_2^2\lambda_n^2 \\
&= \left\|\boldsymbol{c}_n - \boldsymbol{c}\right\|^2 + 2\mu\lambda_n\left[\sum_{i\in\mathcal{I}} f^{(i)}\left(\boldsymbol{c}\right) - \sum_{i\in\mathcal{I}} f^{(i)}\left(\boldsymbol{c}_n\right)\right] \\
&\quad + 2\mu\lambda_n \sum_{i\in\mathcal{I}}\left[f^{(i)}\left(\boldsymbol{c}_n\right) - f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right] + KM_2^2\lambda_n^2.
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$,

(4.7)

$$\begin{aligned}
2\mu\left[\sum_{i\in\mathcal{I}} f^{(i)}\left(\boldsymbol{c}_n\right) - \sum_{i\in\mathcal{I}} f^{(i)}\left(\boldsymbol{c}\right)\right] &\leq \frac{\left\|\boldsymbol{c}_n - \boldsymbol{c}\right\|^2 - \left\|\boldsymbol{c}_{n+1} - \boldsymbol{c}\right\|^2}{\lambda_n} + KM_2^2\lambda_n \\
&\quad + 2\mu\sum_{i\in\mathcal{I}}\left[f^{(i)}\left(\boldsymbol{c}_n\right) - f^{(i)}\left(\boldsymbol{c}_n^{(i-1)}\right)\right].
\end{aligned}$$

Since $(1/\lambda_n)(\|\boldsymbol{c}_n - \boldsymbol{c}\|^2 - \|\boldsymbol{c}_{n+1} - \boldsymbol{c}\|^2) \leq (M_5/\lambda_n)\|\boldsymbol{c}_n - \boldsymbol{c}_{n+1}\|$ $(n \in \mathbb{N})$ and (4.3), we have $\limsup_{n\to\infty}(1/\lambda_n)(\|\boldsymbol{c}_n - \boldsymbol{c}\|^2 - \|\boldsymbol{c}_{n+1} - \boldsymbol{c}\|^2) \leq 0$. Moreover, from $f^{(i)}(\boldsymbol{c}_n) - f^{(i)}(\boldsymbol{c}_n^{(i-1)}) \leq \langle \boldsymbol{c}_n - \boldsymbol{c}_n^{(i-1)}, \nabla f^{(i)}(\boldsymbol{c}_n)\rangle$ $(i \in \mathcal{I}, n \in \mathbb{N})$, we have

$f^{(i)}(\boldsymbol{c}_n) - f^{(i)}(\boldsymbol{c}_n^{(i-1)}) \leq \|\boldsymbol{c}_n - \boldsymbol{c}_n^{(i-1)}\| \|\nabla f^{(i)}(\boldsymbol{c}_n)\|$ $(i \in \mathcal{I}, n \in \mathbb{N})$, which from (4.5) means that $\limsup_{n\to\infty} \sum_{i\in\mathcal{I}} [f^{(i)}(\boldsymbol{c}_n) - f^{(i)}(\boldsymbol{c}_n^{(i-1)})] \leq 0$. Accordingly, (4.7) and (C1) guarantee that, for all $\boldsymbol{c} \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$,

$$\limsup_{n\to\infty} \left[\sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}_n) - \sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}) \right] \leq 0; \text{ i.e., } \limsup_{n\to\infty} \sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}_n) \leq \sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}).$$

Therefore, the convergence of $(\boldsymbol{c}_{n_k})_{k\in\mathbb{N}}$ to $\boldsymbol{c}^* \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$ and the continuity of $\sum_{i\in\mathcal{I}} f^{(i)}$ ensure that, for all $\boldsymbol{c} \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$,

$$\sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}^*) = \lim_{k\to\infty} \sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}_{n_k}) = \limsup_{k\to\infty} \sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}_{n_k}) \leq \sum_{i\in\mathcal{I}} f^{(i)}(\boldsymbol{c}).$$

This implies that $\boldsymbol{c}^* \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$ is the solution to problem (4.1). Since problem (4.1) has a unique solution, denoted by $\boldsymbol{c}^\star$, $(\boldsymbol{c}_{n_k})_{k\in\mathbb{N}}$ converges to the unique solution $\boldsymbol{c}^\star$. Let $\boldsymbol{c}_* \in \mathbb{R}^m$ be an accumulation point of $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$. Then, there exists $(\boldsymbol{c}_{n_l})_{l\in\mathbb{N}}$ $(\subset (\boldsymbol{c}_n)_{n\in\mathbb{N}})$ converging to $\boldsymbol{c}_*$. A discussion similar to the one above leads us to conclude that $\boldsymbol{c}_*$ is the solution to problem (4.1). Accordingly, since any subsequence of $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ converges to $\boldsymbol{c}^\star$, we can conclude that $(\boldsymbol{c}_n)_{n\in\mathbb{N}} = (\boldsymbol{c}_{n-1}^{(K)})_{n\in\mathbb{N}}$ converges to $\boldsymbol{c}^\star$. This implies from (4.5) that $(\boldsymbol{c}_n^{(i-1)})_{n\in\mathbb{N}}$ $(i \in \mathcal{I})$ also converges to $\boldsymbol{c}^\star$. Therefore, $(\boldsymbol{c}_n^{(i)})_{n\in\mathbb{N}}$ $(i \in \mathcal{I})$ generated by Algorithm 4.1 converges to the solution to problem (4.1). □ □

5. Distributed Control under the Profit-Oriented Pricing Scheme

This section presents a broadcast optimization algorithm for solving the following problem that includes Problem 3.2.

$$\text{(5.1)} \qquad \text{Minimize} \sum_{i\in\{0\}\cup\mathcal{I}} f^{(i)}(\boldsymbol{p}) \text{ subject to } \boldsymbol{p} \in \bigcap_{i\in\{0\}\cup\mathcal{I}} \mathrm{Fix}\left(T^{(i)}\right),$$

where $f^{(i)} \colon \mathbb{R}^m \to \mathbb{R}$ $(i \in \{0\} \cup \mathcal{I}, \mathcal{I} := \{1, 2, \ldots, K\})$ is $\alpha^{(i)}$-strongly convex and differentiable, $\nabla f^{(i)} \colon \mathbb{R}^m \to \mathbb{R}^m$ is $L^{(i)}$-Lipschitz continuous, and $T^{(i)} \colon \mathbb{R}^m \to \mathbb{R}^m$ $(i \in \{0\} \cup \mathcal{I})$ is firmly nonexpansive with $\bigcap_{i\in\{0\}\cup\mathcal{I}} \mathrm{Fix}(T^{(i)}) \neq \emptyset$.

Algorithm 5.1 (broadcast optimization algorithm)**.**

Step 0. The operator (peer 0*) sets* $\boldsymbol{p}_0 \in \mathbb{R}^m$ *arbitrarily and transmits* $\boldsymbol{p}_0$ *to all peers.*

Step 1. Given $\boldsymbol{p}_n \in \mathbb{R}^m$*, peer* i $(i \in \{0\} \cup \mathcal{I})$ *computes* $\boldsymbol{p}_{n+1}^{(i)} \in \mathbb{R}^m$ *by*

$$\boldsymbol{p}_{n+1}^{(i)} := T^{(i)}\left(\boldsymbol{p}_n - \mu\lambda_n \nabla f^{(i)}(\boldsymbol{p}_n)\right)$$

and transmits $\boldsymbol{p}_{n+1}^{(i)}$ *to the operator.*

Step 2. The operator computes $\boldsymbol{p}_{n+1} \in \mathbb{R}^m$ *by*

$$\boldsymbol{p}_{n+1} := \frac{1}{K+1} \sum_{i \in \{0\} \cup \mathcal{I}} \boldsymbol{p}_{n+1}^{(i)}$$

and transmits $\boldsymbol{p}_{n+1}$ *to all peers. Put* $n := n+1$*, and go to Step 1.*

Now let us conduct a convergence analysis on Algorithm 5.1.

Theorem 5.1. *Under Assumption 4.1, the sequence* $(\boldsymbol{p}_n)_{n \in \mathbb{N}}$ *generated by Algorithm 5.1 converges to the solution to problem* (5.1)*.*

From Theorem 5.1, Algorithm 5.1 enables the operator to solve Problem 3.2, i.e., problem (5.1) when $f^{(0)} := -\lambda U^{(0)}$, $f^{(i)} := -(1-\lambda) W^{(i)}$ $(i \in \mathcal{I}, \lambda \in (0,1))$, $T^{(0)}$ is defined as in (3.6), and $T^{(i)}$ $(i \in \mathcal{I})$ is defined as in (3.7). Hence, all peers can get the pair of optimal prices $(p_s^\star, p_o^\star)$ in the sense of maximizing the weighted mean of the operator's profit and the social welfare by way of the operator. As a result, peer i $(i \in \mathcal{I})$ can find the pair of optimal storage capacities $(c_s^{(i)\star}, c_o^{(i)\star}) := (d^{(i)}(p_s^\star), s^{(i)}(p_o^\star))$ by using its own supply and demand functions.

Proof. We shall prove that $(\boldsymbol{p}_n)_{n \in \mathbb{N}}$ is bounded. Choose $\boldsymbol{p} \in \bigcap_{i \in \{0\} \cup \mathcal{I}} \mathrm{Fix}(T^{(i)})$ arbitrarily. The nonexpansivity of $T^{(i)}$ and Proposition 3.1 ensure that, for all $i \in \{0\} \cup \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\| \boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p} \right\| &= \left\| T^{(i)} \left(\boldsymbol{p}_n - \mu \lambda_n \nabla f^{(i)} (\boldsymbol{p}_n) \right) - T^{(i)} (\boldsymbol{p}) \right\| \\
&\leq \left\| \left(\boldsymbol{p}_n - \mu \lambda_n \nabla f^{(i)} (\boldsymbol{p}_n) \right) - \boldsymbol{p} \right\| \\
&\leq \left\| \left(\boldsymbol{p}_n - \mu \lambda_n \nabla f^{(i)} (\boldsymbol{p}_n) \right) - \left(\boldsymbol{p} - \mu \lambda_n \nabla f^{(i)} (\boldsymbol{p}) \right) \right\| + \mu \lambda_n \left\| \nabla f^{(i)} (\boldsymbol{p}) \right\| \\
&\leq \left(1 - \tau^{(i)} \lambda_n \right) \| \boldsymbol{p}_n - \boldsymbol{p} \| + \mu \lambda_n \left\| \nabla f^{(i)} (\boldsymbol{p}) \right\| \\
&\leq (1 - \tau \lambda_n) \| \boldsymbol{p}_n - \boldsymbol{p} \| + \mu N_1 \lambda_n,
\end{aligned}$$

where $\tau \leq \tau^{(i)} := 1 - \sqrt{1 - \mu(2\alpha^{(i)} - \mu L^{(i)2})}$ $(i \in \{0\} \cup \mathcal{I})$ and $N_1 := \max_{i \in \{0\} \cup \mathcal{I}} \|\nabla f^{(i)}(\boldsymbol{p})\|$. The definition of $\boldsymbol{p}_n$ means that, for all $n \in \mathbb{N}$,

(5.2)

$$\left\| \boldsymbol{p}_{n+1} - \boldsymbol{p} \right\| = \left\| \frac{1}{K+1} \sum_{i \in \{0\} \cup \mathcal{I}} \left(\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p} \right) \right\| \leq \frac{1}{K+1} \sum_{i \in \{0\} \cup \mathcal{I}} \left\| \boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p} \right\|.$$

Hence, for all $n \in \mathbb{N}$,

$$\left\| \boldsymbol{p}_{n+1} - \boldsymbol{p} \right\| \leq (1 - \tau \lambda_n) \| \boldsymbol{p}_n - \boldsymbol{p} \| + \mu N_1 \lambda_n.$$

A similar argument as in the proof of the boundedness of $(\boldsymbol{c}_n)_{n \in \mathbb{N}}$ in Algorithm 4.1 leads us to conclude that, for all $n \in \mathbb{N}$,

$$\| \boldsymbol{p}_n - \boldsymbol{p} \| \leq \max \left\{ \| \boldsymbol{p}_0 - \boldsymbol{p} \|, \frac{\mu N_1}{\tau} \right\},$$

and hence, $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$ is bounded. Moreover, $(\boldsymbol{p}_n^{(i)})_{n\in\mathbb{N}}$ $(i \in \{0\}\cup\mathcal{I})$ is also bounded from the definition of $\boldsymbol{p}_n$. The Lipschitz continuity of $\nabla f^{(i)}$ implies that $\|\nabla f^{(i)}(\boldsymbol{p}_n) - \nabla f^{(i)}(\boldsymbol{p})\| \leq L^{(i)}\|\boldsymbol{p}_n - \boldsymbol{p}\|$ $(i \in \{0\}\cup\mathcal{I}, n \in \mathbb{N})$, which, from the boundedness of $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$, implies that $(\nabla f^{(i)}(\boldsymbol{p}_n))_{n\in\mathbb{N}}$ $(i \in \{0\}\cup\mathcal{I})$ is bounded.

The nonexpansivity of $T^{(i)}$ $(i \in \{0\}\cup\mathcal{I})$ and Proposition 3.1 guarantee that, for all $i \in \{0\}\cup\mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}_n^{(i)}\right\| &= \left\|T^{(i)}\left(\boldsymbol{p}_n - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right) - T^{(i)}\left(\boldsymbol{p}_{n-1} - \mu\lambda_{n-1}\nabla f^{(i)}\left(\boldsymbol{p}_{n-1}\right)\right)\right\| \\ &\leq \left\|\left(\boldsymbol{p}_n - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right) - \left(\boldsymbol{p}_{n-1} - \mu\lambda_{n-1}\nabla f^{(i)}\left(\boldsymbol{p}_{n-1}\right)\right)\right\| \\ &\leq \left\|\left(\boldsymbol{p}_n - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right) - \left(\boldsymbol{p}_{n-1} - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_{n-1}\right)\right)\right\| \\ &\quad + \mu\left|\lambda_{n-1} - \lambda_n\right|\left\|\nabla f^{(i)}\left(\boldsymbol{p}_{n-1}\right)\right\| \\ &\leq \left(1 - \tau\lambda_n\right)\left\|\boldsymbol{p}_n - \boldsymbol{p}_{n-1}\right\| + \mu N_2\left|\lambda_n - \lambda_{n-1}\right|,\end{aligned}$$

where $N_2 := \max_{i\in\{0\}\cup\mathcal{I}}(\sup_{n\in\mathbb{N}}\|\nabla f^{(i)}(\boldsymbol{p}_n)\|) < \infty$. Summing up the above inequality over all i and going through a similar argument as in (5.2) we find that, for all $n \in \mathbb{N}$,

$$\left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}_n\right\| \leq \left(1 - \tau\lambda_n\right)\left\|\boldsymbol{p}_n - \boldsymbol{p}_{n-1}\right\| + \mu N_2\left|\lambda_n - \lambda_{n-1}\right|.$$

Therefore, in the same manner as in the proof of $\lim_{n\to\infty}\|\boldsymbol{c}_{n+1} - \boldsymbol{c}_n\|/\lambda_n = 0$ (see (4.3)), we find that

$$\lim_{n\to\infty}\frac{\left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}_n\right\|}{\lambda_n} = 0, \quad \lim_{n\to\infty}\left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}_n\right\| = 0. \tag{5.3}$$

Choose $\boldsymbol{p} \in \bigcap_{i\in\{0\}\cup\mathcal{I}}\mathrm{Fix}(T^{(i)})$ arbitrarily; i.e., $\boldsymbol{p} = T^{(i)}(\boldsymbol{p})$ $(i \in \{0\}\cup\mathcal{I})$. From $\boldsymbol{p}_{n+1}^{(i)} := T^{(i)}(\boldsymbol{p}_n - \mu\lambda_n\nabla f^{(i)}(\boldsymbol{p}_n))$, the firm nonexpansivity of $T^{(i)}$, and Proposition 3.3, we have that, for all $i \in \{0\}\cup\mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}&\left\|\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}\right\|^2 \\ &\leq \left\|\left(\boldsymbol{p}_n - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right) - \boldsymbol{p}\right\|^2 - \left\|\left(\left(\boldsymbol{p}_n - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right) - \boldsymbol{p}\right) - \left(\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}\right)\right\|^2 \\ &= \left\|\left(\boldsymbol{p}_n - \boldsymbol{p}\right) - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right\|^2 - \left\|\left(\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}\right) - \mu\lambda_n\nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right\|^2.\end{aligned}$$

From $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 - 2\langle\boldsymbol{x}, \boldsymbol{y}\rangle + \|\boldsymbol{y}\|^2$ $(\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m)$, we find that

$$\begin{aligned}\left\|\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}\right\|^2 &\leq \left\|\boldsymbol{p}_n - \boldsymbol{p}\right\|^2 - 2\mu\lambda_n\left\langle\boldsymbol{p}_n - \boldsymbol{p}, \nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right\rangle - \left\|\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}\right\|^2 \\ &\quad + 2\mu\lambda_n\left\langle\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}, \nabla f^{(i)}\left(\boldsymbol{p}_n\right)\right\rangle \\ &\leq \left\|\boldsymbol{p}_n - \boldsymbol{p}\right\|^2 - \left\|\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}\right\|^2 + N_3\lambda_n,\end{aligned} \tag{5.4}$$

where $N_3 := \max_{i \in \{0\} \cup \mathcal{I}} (\sup_{n\in\mathbb{N}} 2\mu |\langle \boldsymbol{p} - \boldsymbol{p}_{n+1}^{(i)}, \nabla f^{(i)}(\boldsymbol{p}_n)\rangle|) < \infty$. Since the convexity of $\|\cdot\|^2$ ensures that, for all $n \in \mathbb{N}$,

(5.5)
$$\left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}\right\|^2 = \left\| \frac{1}{K+1} \sum_{i\in\{0\}\cup\mathcal{I}} \left(\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}\right)\right\|^2 \leq \frac{1}{K+1} \sum_{i\in\{0\}\cup\mathcal{I}} \left\|\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}\right\|^2,$$

summing up (5.4) over all i implies that, for all $n \in \mathbb{N}$,

$$\left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}\right\|^2 \leq \|\boldsymbol{p}_n - \boldsymbol{p}\|^2 - \frac{1}{K+1} \sum_{i\in\{0\}\cup\mathcal{I}} \left\|\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}\right\|^2 + N_3\lambda_n.$$

Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{K+1} \sum_{i\in\{0\}\cup\mathcal{I}} \left\|\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}\right\|^2 &\leq \|\boldsymbol{p}_n - \boldsymbol{p}\|^2 - \left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}\right\|^2 + N_3\lambda_n \\ &= \left(\|\boldsymbol{p}_n - \boldsymbol{p}\| + \left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}\right\|\right)\left(\|\boldsymbol{p}_n - \boldsymbol{p}\| - \left\|\boldsymbol{p}_{n+1} - \boldsymbol{p}\right\|\right) \\ &\quad + N_3\lambda_n \\ &\leq N_4 \left\|\boldsymbol{p}_n - \boldsymbol{p}_{n+1}\right\| + N_3\lambda_n,\end{aligned}$$

where $N_4 := \sup_{n\in\mathbb{N}}(\|\boldsymbol{p}_n - \boldsymbol{p}\| + \|\boldsymbol{p}_{n+1} - \boldsymbol{p}\|) < \infty$. From (C1) and (5.3), we find that $\lim_{n\to\infty} \|\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}\| = 0$ $(i \in \{0\} \cup \mathcal{I})$. The nonexpansivity of $T^{(i)}$ $(i \in \{0\} \cup \mathcal{I})$ implies that, for all $i \in \{0\} \cup \mathcal{I}$ and for all $n \in \mathbb{N}$, $\|\boldsymbol{p}_{n+1}^{(i)} - T^{(i)}(\boldsymbol{p}_n)\| \leq \|T^{(i)}(\boldsymbol{p}_n - \mu\lambda_n \nabla f^{(i)}(\boldsymbol{p}_n)) - T^{(i)}(\boldsymbol{p}_n)\| \leq \mu\lambda_n \|\nabla f^{(i)}(\boldsymbol{p}_n)\| \leq \mu N_2 \lambda_n$. Accordingly, (C1) means that $\lim_{n\to\infty} \|\boldsymbol{p}_{n+1}^{(i)} - T^{(i)}(\boldsymbol{p}_n)\| = 0$ $(i \in \{0\} \cup \mathcal{I})$. Hence, from $\|\boldsymbol{p}_n - T^{(i)}(\boldsymbol{p}_n)\| \leq \|\boldsymbol{p}_n - \boldsymbol{p}_{n+1}^{(i)}\| + \|\boldsymbol{p}_{n+1}^{(i)} - T^{(i)}(\boldsymbol{p}_n)\|$ $(i \in \{0\} \cup \mathcal{I}, n \in \mathbb{N})$, we have

$$\lim_{n\to\infty} \left\|\boldsymbol{p}_n - T^{(i)}(\boldsymbol{p}_n)\right\| = 0 \;\; (i \in \{0\} \cup \mathcal{I}). \tag{5.6}$$

The boundedness of $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$ guarantees the existence of an accumulation point of $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$. Let $\boldsymbol{p}^* \in \mathbb{R}^m$ be an arbitrary accumulation point of $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$. Accordingly, a subsequence $(\boldsymbol{p}_{n_k})_{k\in\mathbb{N}}$ of $(\boldsymbol{p}_n)_{n\in\mathbb{N}}$ exists such that $(\boldsymbol{p}_{n_k})_{k\in\mathbb{N}}$ converges to $\boldsymbol{p}^*$. In the same manner as in the proof of $\boldsymbol{c}^* \in \bigcap_{i\in\mathcal{I}} \mathrm{Fix}(T^{(i)})$ in Section 4, the continuity of $T^{(i)}$ $(i \in \{0\} \cup \mathcal{I})$ and (5.6) guarantee that $\boldsymbol{p}^* \in \bigcap_{i\in\{0\}\cup\mathcal{I}} \mathrm{Fix}(T^{(i)})$.

From the nonexpansivity of $T^{(i)}$ $(i \in \{0\} \cup \mathcal{I})$, we find that, for all $i \in \{0\} \cup \mathcal{I}$ and for all $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}\right\|^2 &= \left\|T^{(i)}\left(\boldsymbol{p}_n - \mu\lambda_n \nabla f^{(i)}(\boldsymbol{p}_n)\right) - T^{(i)}(\boldsymbol{p})\right\|^2 \\ &\leq \left\|(\boldsymbol{p}_n - \boldsymbol{p}) - \mu\lambda_n \nabla f^{(i)}(\boldsymbol{p}_n)\right\|^2 \\ &= \|\boldsymbol{p}_n - \boldsymbol{p}\|^2 - 2\mu\lambda_n \left\langle \boldsymbol{p}_n - \boldsymbol{p}, \nabla f^{(i)}(\boldsymbol{p}_n)\right\rangle + \mu^2\lambda_n^2 \left\|\nabla f^{(i)}(\boldsymbol{p}_n)\right\|^2,\end{aligned}$$

which from the differentiability of $f^{(i)}$ $(i \in \{0\} \cup \mathcal{I})$ implies that

$$\left\|\boldsymbol{p}_{n+1}^{(i)} - \boldsymbol{p}\right\|^2 \leq \|\boldsymbol{p}_n - \boldsymbol{p}\|^2 + 2\mu\lambda_n \left[f^{(i)}(\boldsymbol{p}) - f^{(i)}(\boldsymbol{p}_n)\right] + \mu^2 N_2^2 \lambda_n^2.$$

Summing up the above inequality over all i and (5.5) lead to

$$\|\boldsymbol{p}_{n+1} - \boldsymbol{p}\|^2 \leq \|\boldsymbol{p}_n - \boldsymbol{p}\|^2 + \frac{2\mu\lambda_n}{K+1} \sum_{i \in \{0\}\cup\mathcal{I}} \left[f^{(i)}(\boldsymbol{p}) - f^{(i)}(\boldsymbol{p}_n)\right] + \mu^2 N_2^2 \lambda_n^2$$

for all $n \in \mathbb{N}$, which means

$$\begin{aligned}\frac{2\mu}{K+1} \sum_{i \in \{0\}\cup\mathcal{I}} \left[f^{(i)}(\boldsymbol{p}_n) - f^{(i)}(\boldsymbol{p})\right] &\leq \frac{\|\boldsymbol{p}_n - \boldsymbol{p}\|^2 - \|\boldsymbol{p}_{n+1} - \boldsymbol{p}\|^2}{\lambda_n} + \mu^2 N_2^2 \lambda_n \\ &\leq \frac{N_4 \|\boldsymbol{p}_{n+1} - \boldsymbol{p}_n\|}{\lambda_n} + \mu^2 N_2^2 \lambda_n\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, (C1) and (5.3) guarantee that, for all $\boldsymbol{p} \in \bigcap_{i \in \{0\}\cup\mathcal{I}} \mathrm{Fix}(T^{(i)})$,

$$\limsup_{n \to \infty} \sum_{i \in \{0\}\cup\mathcal{I}} \left[f^{(i)}(\boldsymbol{p}_n) - f^{(i)}(\boldsymbol{p})\right] \leq 0. \tag{5.7}$$

In the same manner as in the proof of Theorem 4.1 and (5.7), we find that $(\boldsymbol{p}_n)_{n \in \mathbb{N}}$ generated by Algorithm 5.1 converges to the solution to problem (5.1). □ □

6. Numerical Examples

We conducted numerical experiments comparing the capabilities of the proposed algorithms with different parameters for solving Problem 3.1 and 3.2 when $K = 100$. We used $\mu = 10^{-1}, 10^{-3}$ and $\lambda_n := 1/(n+1)^{0.45}$. We randomly chose $a^{(i)}, b^{(i)} \in (0, 5]$, $p_{\min}^{(i)} \in [0, 10]$, $p_{\max}^{(i)} \in [90, 100]$ $(i = 1, 2, \ldots, 100)$. The computer used in the experiment had an Intel Boxed Core i7 i7-870 2.93 GHz 8M CPU and 8 GB of memory. The language was MATLAB 7.13.

In the experiment, we set $\boldsymbol{c} := \boldsymbol{c}_0 = \boldsymbol{c}^{(i)}$ $(i \in \mathcal{I})$ in Algorithm 4.1, selected one hundred random points $\boldsymbol{c} = \boldsymbol{c}(k)$ $(k = 1, 2, \ldots, 100)$, and executed the algorithm on these points. Let $\boldsymbol{c}(k)$ $(\in \mathbb{R}^{100} \times \mathbb{R}^{100})$ be one of the randomly selected points and let $(\boldsymbol{c}_n(k))_{n \in \mathbb{N}}$ $(\subset \mathbb{R}^{100} \times \mathbb{R}^{100})$ be the sequence generated by $\boldsymbol{c}(k)$ and Algorithm 4.1. We employed $D_n(k) := \|\boldsymbol{c}_n(k) - T^{(100)} T^{(99)} \cdots T^{(1)}(\boldsymbol{c}_n(k))\|$ $(k = 1, 2, \ldots, 100, n \in \mathbb{N})$ and their mean value, $D_n := (1/100) \sum_{k=1}^{100} D_n(k)$ $(n \in \mathbb{N})$, where $T^{(i)}$ $(i \in \mathcal{I})$ is defined as in (3.3). If $(D_n)_{n \in \mathbb{N}}$ converges to 0, Algorithm 4.1 converges to a point in $\bigcap_{i \in \mathcal{I}} \mathrm{Fix}(T^{(i)})$.

Figure 1 describes the behaviors of D_n for Algorithm 4.1 when $\mu = 10^{-1}$ and 10^{-3}. Here, $(D_n)_{n \in \mathbb{N}}$ converges to 0; i.e., Algorithm 4.1 converges to a point in the constraint set in Problem 3.1. In particular, it converges quickly when $\mu = 10^{-3}$. This is because $\mu\lambda_n \in (0, \min_{i\in\mathcal{I}} 2L^{(i)}/{\alpha^{(i)}}^2)$, which is the convergence condition of the algorithm, is satisfied in the early stages (see Assumption 4.1). Let us define $c_{n,s}^{(i)} := (1/100)\sum_{k=1}^{100} c_{n,s}^{(i)}(k)$ and $c_{n,o}^{(i)} := (1/100)\sum_{k=1}^{100} c_{n,o}^{(i)}(k)$ $(i \in \mathcal{I}, n \in \mathbb{N})$, where $\boldsymbol{c}_n(k) := (\boldsymbol{c}_{n,s}(k), \boldsymbol{c}_{n,o}(k)) \in \mathbb{R}^{100} \times \mathbb{R}^{100}$, $\boldsymbol{c}_{n,s}(k) := (c_{n,s}^{(1)}(k), c_{n,s}^{(2)}(k), \ldots, c_{n,s}^{(100)}(k))^T \in \mathbb{R}^{100}$, $\boldsymbol{c}_{n,o}(k) := (c_{n,o}^{(1)}(k), c_{n,o}^{(2)}(k), \ldots, c_{n,o}^{(100)}(k))^T \in \mathbb{R}^{100}$ $(k = 1, 2, \ldots, 100, n \in \mathbb{N})$. Figures 2–4 show the behaviors of $c_{n,s}^{(i)}$ and $c_{n,o}^{(i)}$ $(i = 20, 40, 60)$ generated by Algorithm 4.1 with $\mu = 10^{-3}$. We can see from these figures that the convergent point $\boldsymbol{c}^\star := (\boldsymbol{c}_s^\star, \boldsymbol{c}_o^\star)$ satisfies ${c_o^{(i)}}^\star = {c_s^{(i)}}^\star$ $(i = 20, 40, 60)$; i.e., in the symmetric scheme, peer i offers ${c_o^{(i)}}^\star$ equivalent to its own used amount ${c_s^{(i)}}^\star$.

Next, we solved Problem 3.2 with $\lambda := 1/2$ by using Algorithm 5.1. We selected one hundred random points $\boldsymbol{p}_0 = \boldsymbol{p}(k)$ $(k = 1, 2, \ldots, 100)$ and executed the algorithm on these points. Let $\boldsymbol{p}(k)$ $(\in \mathbb{R} \times \mathbb{R})$ be one of the randomly selected points, and let $(\boldsymbol{p}_n(k))_{n\in\mathbb{N}} := (p_{n,s}(k), p_{n,o}(k))$ $(\subset \mathbb{R} \times \mathbb{R})$ be the sequence generated by $\boldsymbol{p}(k)$ and Algorithm 5.1. We employed $d_n(k) := \|\boldsymbol{p}_n(k) - T^{(100)}T^{(99)}\cdots T^{(0)}(\boldsymbol{p}_n(k))\|$ $(k = 1, 2, \ldots, 100, n \in \mathbb{N})$ and $d_n := (1/100)\sum_{k=1}^{100} d_n(k)$ $(n \in \mathbb{N})$, where $T^{(0)}$ and $T^{(i)}$ $(i \in \mathcal{I})$ are defined as in (3.6) and (3.7). We also employed $p_{n,s} := (1/100)\sum_{k=1}^{100} p_{n,s}(k)$ and $p_{n,o} := (1/100)\sum_{k=1}^{100} p_{n,o}(k)$ $(n \in \mathbb{N})$.

Figure 5 indicates the behavior of d_n for Algorithm 5.1 when $\mu = 10^{-1}$ and 10^{-3}. Since $(d_n)_{n\in\mathbb{N}}$ converges to 0, we find that Algorithm 5.1 converges to a point in the constrained set in Problem 3.2. As we pointed out in the above paragraph (Figure 1), Algorithm 5.1 with $\mu = 10^{-3}$ converges faster than it does with $\mu = 10^{-1}$. Figure 6 shows the behavior of $p_{n,s}$ and $p_{n,o}$ for Algorithm 5.1 when $\mu = 10^{-3}$, and the $(p_{n,s})_{n\in\mathbb{N}}$ and $(p_{n,o})_{n\in\mathbb{N}}$ converge to the same point. This implies that the optimal prices, $p_s^\star$ and $p_o^\star$, for maximizing the mean of the operator's profit and social welfare are approximately the same.

7. Conclusion

We discussed the storage allocation problems caused by incentive schemes (the symmetric scheme and the profit-oriented pricing scheme) for controlling P2P data storage systems. We presented two distributed optimization algorithms, called the incremental gradient algorithm and the broadcast optimization algorithm, for solving them and performed convergence analyses. The incremental gradient algorithm can be applied to the symmetric scheme, while the broadcast optimization algorithm is for the profit-oriented pricing scheme. We gave numerical results showing that the algorithms converge to solutions to the storage allocation problems.

Acknowledgments I am sincerely grateful to the Guest Editor, Shigeo Akashi, and the anonymous reviewers for helping me improve the original manuscript submitted to Special Issue-Dedicated to Wataru Takahashi on the occasion of his 70th birth day.

This work was supported by the Japan Society for the Promotion of Science through a Grant-in-Aid for Scientific Research (C) (15K04763).

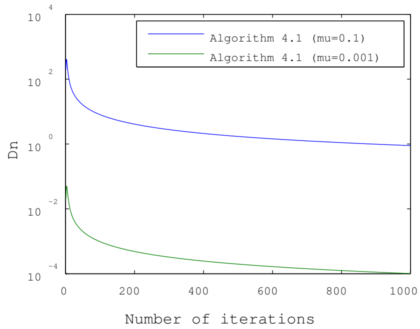

Figure 1. Behavior of D_n for Algorithm 4.1 when $\mu = 10^{-1}, 10^{-3}$

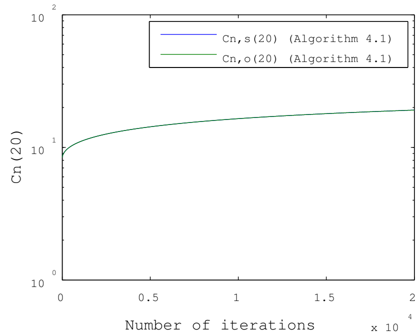

Figure 2. Behavior of $c_{n,s}^{(20)}$ and $c_{n,o}^{(20)}$ for Algorithm 4.1 when $\mu = 10^{-3}$

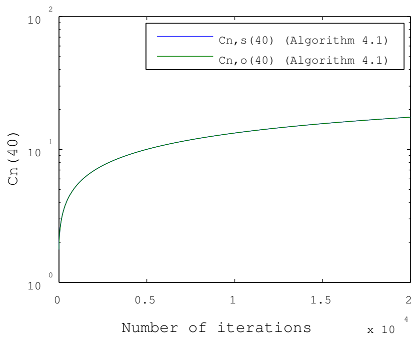

Figure 3. Behavior of $c_{n,s}^{(40)}$ and $c_{n,o}^{(40)}$ for Algorithm 4.1 when $\mu = 10^{-3}$

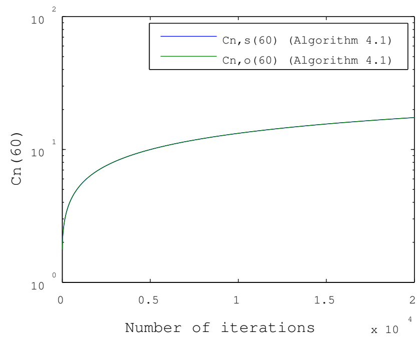

Figure 4. Behavior of $c_{n,s}^{(60)}$ and $c_{n,o}^{(60)}$ for Algorithm 4.1 when $\mu = 10^{-3}$

References

[1] Antoniadis, P., Courcoubetis, C., Mason, R.: Comparing economic incentives in peer-to-peer networks. Computer Networks **46**, 133–146 (2004)

[2] Bauschke, H.H., Borwein, J.M.: On projection algorithms for solving convex feasibility problems. SIAM Review **38**(3), 367–426 (1996)

[3] Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer (2011)

[4] Berinde, V.: Iterative Approximation of Fixed Points. Springer (2007)

[5] Bertsekas, D.P., El Baz, D.: Distributed asynchronous relaxation methods for convex network flow problems. SIAM Journal on Control and Optimization **25**, 74–85 (1987)

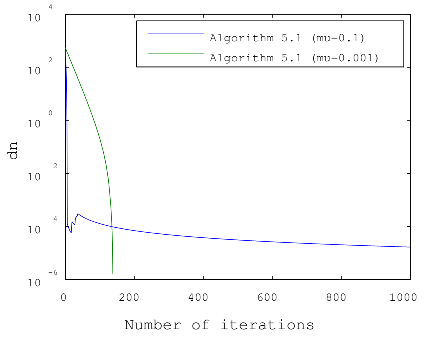

FIGURE 5. Behavior of d_n for Algorithm 5.1 when $\mu = 10^{-1}, 10^{-3}$

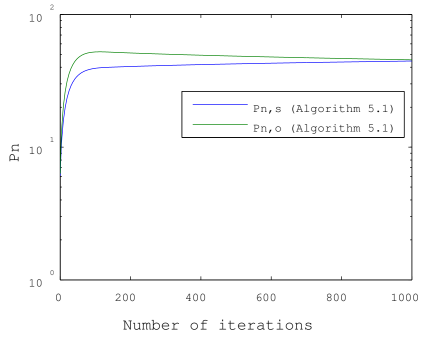

FIGURE 6. Behavior of $p_{n,s}$ and $p_{n,o}$ for Algorithm 5.1 when $\mu = 10^{-3}$

[6] Bertsekas, D.P., Hosein, P.A., Tseng, P.: Relaxation methods for network flow problems with convex arc costs. SIAM Journal on Control and Optimization **25**, 1219–1243 (1987)

[7] Bertsekas, D.P., Nedić, A., Ozdaglar, A.E.: Convex Analysis and Optimization. Athena Scientific (2003)

[8] Blatt, D., Hero, A.O., Gauchman, H.: A convergent incremental gradient method with a constant step size. SIAM Journal on Optimization **18**, 29–51 (2007)

[9] Boyd, S., Parikh, N., Chu, E., Peleato, B., Eckstein, J.: Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends in Machine Learning **3**, 1–122 (2011)

[10] Chen, S., Li, X., Zhou, X.Y.: Stochastic linear quadratic regulators with indefinite control weight costs. SIAM Journal on Control and Optimization **36**, 1685–1702 (1998)

[11] Combettes, P.L.: Iterative construction of the resolvent of a sum of maximal monotone operators. Journal of Convex Analysis **16**, 727–748 (2009)

[12] Combettes, P.L., Pesquet, J.C.: A proximal decomposition method for solving convex variational inverse problems. Inverse Problems **24**, article ID 065,014 (2008)

[13] Courcoubetis, C., Weber, R.: Incentives for large peer-to-peer systems. IEEE Journal on Selected Areas in Communications **24**, 1034–1050 (2006)

[14] Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press (1990)

[15] Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker (1984)

[16] Iiduka, H.: Fixed point optimization algorithms for distributed optimization in networked systems. SIAM Journal on Optimization **23**, 1–26 (2013)

[17] Iiduka, H., Yamada, I.: Computational method for solving a stochastic linear-quadratic control problem given an unsolvable stochastic algebraic Riccati equation. SIAM Journal on Control and Optimization **50**, 2173–2192 (2012)

[18] Johansson, B., Rabi, M., Johansson, M.: A randomized incremental subgradient method for distributed optimization in networked systems. SIAM Journal on Optimization **20**, 1157–1170 (2009)

[19] Kiwiel, K.C.: Convergence of approximate and incremental subgradient methods for convex optimization. SIAM Journal on Optimization **14**, 807–840 (2004)

[20] Low, S., Lapsley, D.E.: Optimization flow control, i: Basic algorithm and convergence. IEEE/ACM Transactions on Networking **7**(6), 861–874 (1999)

[21] Maillé, P., Toka, L.: Managing a peer-to-peer data storage system in a selfish society. IEEE Journal on Selected Areas in Communications **26**(7), 1295–1301 (2008)

[22] Nedić, A., Bertsekas, D.P.: Incremental sugradient methods for nondifferentiable optimization. SIAM Journal on Optimization **12**, 109–138 (2001)

[23] Nedić, A., Olshevsky, A., Ozdaglar, A., Tsitsiklis, J.N.: On distributed averaging algorithms and quantization effects. IEEE Transactions on Automatic Control **54**, 2506–2517 (2009)

[24] Nedić, A., Ozdaglar, A.: Distributed subgradient methods for multi agent optimization. IEEE Transactions on Automatic Control **54**, 48–61 (2009)

[25] Nesterov, Y.: Primal-dual subgradient methods for convex problems. Mathematical Programming **120**, 261–283 (2009)

[26] Rami, M.A., Zhou, X.Y.: Linear matrix inequalities, riccati equations, and indefinite stochastic linear quadratic controls. IEEE Transactions on Automatic Control **45**, 1131–1142 (2000)

[27] Sharma, S., Teneketzis, D.: An externalities-based decentralized optimal power allocation algorithm for wireless networks. IEEE/ACM Transactions on Networking **17**, 1819–1831 (2009)

[28] Srikant, R.: Mathematics of Internet Congestion Control. Birkhauser (2004)

[29] Tseng, P., Bertsekas, D.P., Tsitsiklis, J.N.: Partially asynchronous, parallel algorithms for network flow and other problems. SIAM Journal on Control and Optimization **28**, 678–710 (1990)

[30] Xiao, L., Boyd, S., Kim, S.J.: Distributed average consensus with least-mean-square deviation. Journal of Parallel and Distributed Computing **67**, 33–46 (2007)

[31] Yamada, I.: The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings. In: D. Butnariu, Y. Censor, S. Reich (eds.) Inherently Parallel Algorithms for Feasibility and Optimization and Their Applications, pp. 473–504. Elsevier (2001)

(H. Iiduka) Department of Computer Science, Meiji University, 1-1-1 Higashimita, Tama-ku, Kawasaki-shi, Kanagawa, 214-8571 Japan, Japan

E-mail address: `iiduka@cs.meiji.ac.jp`