Convex Optimization over Fixed Point Sets of Quasi-nonexpansive and Nonexpansive Mappings in Utility-Based Bandwidth Allocation Problems with Operational Constraints

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Abstract

Network bandwidth allocation is a central issue in modern communication networks. The main objective of the bandwidth allocation is to allocate an optimal bandwidth for maximizing a predefined utility over the capacity constraints to traffic sources. When a centralized operator, which manages all the bandwidth allocations in the network, has a certain operational policy, the bandwidth allocation reflecting the operational policy should result in the network being more stable and reliable. Accordingly, we need to solve a network bandwidth allocation problem under both capacity constraints and operational constraints. To develop a novel algorithm for solving the problem, we translate the network bandwidth allocation problem into one of minimizing a convex objective function over the intersection of the fixed point sets of certain quasi-nonexpansive and nonexpansive mappings and propose a fixed point optimization algorithm for solving it. We numerically compare the proposed algorithm with the existing algorithm for solving a concrete bandwidth allocation problem and show its effectiveness.

Keywords: utility-based bandwidth allocation problem, quasi-nonexpansive mapping, nonexpansive mapping, fixed point optimization algorithm
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1. Introduction

1.1. Background

Network resource allocation is needed for making communication networks reliable and stable, and it is of practical importance to allocate, fairly and effectively, finite network resources, such as power [15, 29], channel [22], and bandwidth [16, 19, 23, 30, 36], to network users.

The objective of *utility-based bandwidth allocation* [23, 30, 36] in particular is to share the available bandwidth among traffic sources so as to maximize the overall utility under the capacity constraints.

The utility is modeled as a function, denoted by $\mathcal{U}$, of the transmission rates allocated to the traffic sources, and it represents the efficiency and fairness of bandwidth sharing [23, 30, 36]. We assume that $\mathcal{U}$ is continuously differentiable and concave. A well-known utility function is the *weighted proportionally fair* function [23, 30, 36] defined for all $x := (x_1, x_2, \ldots, x_S)^T \in \mathbb{R}^S_+ \backslash \{0\}$ by $\mathcal{U}_{\mathrm{pf}}(x) := \sum_{s \in \mathcal{S}} w_s \log x_s$, where x_s (> 0) denotes the transmission rate of source s $(\in \mathcal{S} := \{1, 2, \ldots, S\})$, w_s (> 0) stands for the weighted parameter for source s, and $\mathbb{R}^S_+ := \{(x_1, x_2, \ldots, x_S)^T \in \mathbb{R}^S \colon x_s \geq 0 \ (s \in \mathcal{S})\}$. The optimal bandwidth allocation corresponding to $\mathcal{U}_{\mathrm{pf}}$ is said to be *weighted proportionally fair*.

The capacity constraint for each link is an inequality constraint in which the sum of the transmission rates of all the sources sharing the link is less than or equal to the capacity of the link, and hence, the capacity constraint set for each link l $(\in \mathcal{L} := \{1, 2, \ldots, L\})$ is expressed as $\mathbb{R}^S_+ \cap C_l$, where

$$C_l := \left\{ x := (x_1, x_2, \ldots, x_S)^T \in \mathbb{R}^S \colon \sum_{s \in \mathcal{S}} x_s I_{s,l} \leq c_l \right\},$$

c_l (> 0) stands for the capacity of link l, and $I_{s,l}$ takes the value 1 if l is the link used by source s, and 0 otherwise.

Therefore, our objective in bandwidth allocation is to solve the following *utility-based bandwidth allocation problem* [23], [36, Chapter 2] for maximizing the utility function subject to the capacity constraints:

$$\text{Maximize } \mathcal{U}_{\mathrm{pf}}(x) \text{ subject to } x \in C,$$

where C ($\subset \mathbb{R}^S$) stands for the capacity constraint set defined by

$$C := \mathbb{R}^S_+ \cap \bigcap_{l \in \mathcal{L}} C_l = \mathbb{R}^S_+ \cap \bigcap_{l \in \mathcal{L}} \left\{ (x_1, x_2, \ldots, x_S)^T \in \mathbb{R}^S \colon \sum_{s \in \mathcal{S}} x_s I_{s,l} \leq c_l \right\}. \tag{1}$$

1.2. Utility-based bandwidth allocation problem with operational constraint

We will discuss a utility-based bandwidth allocation problem subject to not only the capacity constraints but also an operational constraint. The operator has an operational policy to make the network more stable and reliable. For example, when sources exist in the network such that they get a low (resp. high) degree of satisfaction, the operator attempts to re-allocate bandwidth so as to enable them to send data at high (resp. low) transmission rates. When the available bandwidth is limited in the network, the operator needs to control the sum of the transmission rates of all sources. When the network is controlled by using a certain indicator function which represents the network's performance, the operator tries to design the network so as to satisfy a constraint incorporating the indicator function. The operational constraint set representing such operational policies can be written as

$$C_{\text{op}} := \left\{ x := (x_1, x_2, \ldots, x_S)^T \in \mathbb{R}^S \colon \mathcal{P}(x) \leq p \right\}, \tag{2}$$

where $\mathcal{P} \colon \mathbb{R}^S \to \mathbb{R}$ is convex (i.e., $\mathcal{P}$ satisfies the continuity [5, Theorem 4.1.3]) and is not always differentiable, and $p \in \mathbb{R}$. The operator can set $C_{\text{op}} = \{x \in \mathbb{R}^S \colon x_{s_0} \leq p\}$ when it tries to limit the transmission rate of source s_0, $C_{\text{op}} = \{x \in \mathbb{R}^S \colon \sum_{s \in \mathcal{S}} \omega_s x_s \leq p\}$ ($\omega_s \geq 0$ ($s \in \mathcal{S}$)) when it tries to limit the transmission rates of all sources, and $C_{\text{op}} = \{x \in \mathbb{R}^S \colon \sum_{s \in \mathcal{S}} \omega_s \mathcal{P}_s(x_s) \leq p\}$ ($\omega_s \geq 0$ ($s \in \mathcal{S}$), $\mathcal{P}_s \colon \mathbb{R} \to \mathbb{R}$ is nondifferentiable[1]) when the network is controlled by $\mathcal{P}(x) := \sum_{s \in \mathcal{S}} \omega_s \mathcal{P}_s(x_s)$.

Therefore, we can formulate a utility-based bandwidth allocation problem with both the capacity constraints and the operational constraint as follows:

$$\text{Maximize } \mathcal{U}_{\text{pf}}(x) \text{ subject to } x \in C \cap C_{\text{op}}, \tag{3}$$

where one assumes $C \cap C_{\text{op}} \neq \emptyset$.[2]

[1] If the network's performance increases when all sources' transmission rates are more than a certain value x^0 (> 0), $\mathcal{P}_s(x)$ is expressed as 0 ($0 \leq x \leq x^0$), or $x + x^0$ ($x \geq x^0$).

[2] For example, $0 \in C \cap C_{\text{op}}$ holds when $\mathcal{P}(x) := \sum_{s \in \mathcal{S}} \omega_s x_s$, $\omega_s \in \mathbb{R}$ ($s \in \mathcal{S}$), and $p \geq 0$. Since the operator knows the explicit form of C, it can set C_{op} such that $C \cap C_{\text{op}} \neq \emptyset$.

There are useful methods [7, 10, 11, 25, 31, 37, 42] for solving optimization problems with nonsmooth constraints and optimization problems with nonsmooth objective functions. One avenue for addressing the lack of smoothness is via a variety of smoothing techniques (e.g., deterministic smoothing techniques [7] and convolution-based smoothing techniques [42]). Other methods are, for example, path search algorithms [11, Subchapter 8.1], trust region methods [11, Subchapter 8.4], equation-based algorithms [11, Chapter 9], variational inequality-based algorithms [11, Chapter 10], subgradient methods [31, Chapter 3], and bundle trust region algorithms [31, Subchapter 3.3].

To develop an algorithm for solving Problem (3), we will focus on the following *variational inequality* [9, Chapter II], [10, Chapter 1], [24, Chapter I], [33, Subchapter 6.D] which coincides with Problem (3) [9, Chapter 2, Proposition 2.1 (2.1) and (2.2)].

Problem 1.1 (Utility-based bandwidth allocation problem under capacity constraints)**.**

$$\begin{aligned} \textit{Find } x^\star \in & \mathrm{VI}\,(C \cap C_{\mathrm{op}}, -\nabla \mathcal{U}_{\mathrm{pf}}) \\ & := \left\{ x^\star \in C \cap C_{\mathrm{op}} \colon \langle x - x^\star, -\nabla \mathcal{U}_{\mathrm{pf}}(x^\star) \rangle \geq 0 \ (x \in C \cap C_{\mathrm{op}}) \right\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ *stands for the inner product of* $\mathbb{R}^S$ *and* $\nabla \mathcal{U}_{\mathrm{pf}} \colon \mathbb{R}^S \to \mathbb{R}^S$ *is the gradient of* $\mathcal{U}_{\mathrm{pf}}$.

In this paper, we shall devise an iterative algorithm for solving Problem 1.1 based on iterative techniques for *optimization over the fixed point sets* of certain mappings. With this goal in mind, we will translate Problem 1.1 into an optimization problem over the intersection of the fixed point sets.

1.3. Optimization problem over fixed point sets

We first show that C in (1) can be expressed as the fixed point set of a mapping composed of the metric projections onto C_ls. Let us define the following mapping, $T_{\mathrm{proj}} \colon \mathbb{R}^S \to \mathbb{R}^S$, composed of the metric projections onto $\mathbb{R}^S_+$ and C_ls:

$$T_{\mathrm{proj}} := P_{\mathbb{R}^S_+} \prod_{l \in \mathcal{L}} P_{C_l} = P_{\mathbb{R}^S_+} P_{C_1} P_{C_2} \cdots P_{C_L}, \tag{4}$$

where P_D stands for the metric projection[3] onto a nonempty, closed convex set D ($\subset \mathbb{R}^S$). Then, T_{proj} satisfies the *nonexpansivity* condition[4] because $P_{\mathbb{R}^S_+}$ and P_{C_l}s are nonexpansive. Moreover, C in (1) coincides with the fixed point set of T_{proj} [1, Proposition 2.10], i.e.,

$$C = \text{Fix}(T_{\text{proj}}) := \left\{ x \in \mathbb{R}^S \colon T_{\text{proj}}(x) = x \right\}. \tag{5}$$

Let us show that C_{op} in (2) can be expressed as the fixed point set of a *subgradient projection*. The subgradient projection relative to $\mathcal{P}(\cdot) - p$, denoted by $Q_{\text{sp}} \colon \mathbb{R}^S \to \mathbb{R}^S$, is defined as follows:

$$Q_{\text{sp}}(x) := \begin{cases} x - \dfrac{\mathcal{P}(x) - p}{\|\mathcal{P}'(x)\|^2} \mathcal{P}'(x) & \text{if } \mathcal{P}(x) > p, \\ x & \text{otherwise,} \end{cases} \tag{6}$$

where $\mathcal{P}'(x) \in \partial \mathcal{P}(x) := \{ z \in \mathbb{R}^S \colon (\mathcal{P}(y) - p) \geq (\mathcal{P}(x) - p) + \langle y - x, z \rangle \ (y \in \mathbb{R}^S) \}$ stands for the subgradient of $\mathcal{P}(\cdot) - p$ at $x \in \mathbb{R}^S$. Accordingly, Q_{sp} is *quasi-firmly nonexpansive*[5] [3, Proposition 2.3], [38, Lemma 2.8 in Subchapter 2.4.3]. Moreover, Q_{sp} is *fixed-point closed*[6] [2, Lemma 3.1(ii)]. $\text{Fix}(Q_{\text{sp}})$ is characterized as follows [3, Proposition 2.3], [38, Lemma 2.8 in Subchapter 2.4.3]:

$$C_{\text{op}} = \text{Fix}(Q_{\text{sp}}) := \left\{ x \in \mathbb{R}^S \colon Q_{\text{sp}}(x) = x \right\}. \tag{7}$$

Meanwhile, $-\nabla \mathcal{U}_{\text{pf}}$ is *strongly monotone and Lipschitz continuous on a certain set.*[7] From (5) and (7), we can see that Problem 1.1 can be expressed as the following variational inequality over the intersection of the fixed point sets of T_{proj} and Q_{sp} defined by (4) and (6), respectively:

$$\text{Find } x^\star \in \text{VI}\left(\text{Fix}\left(T_{\text{proj}}\right) \cap \text{Fix}\left(Q_{\text{sp}}\right), -\nabla \mathcal{U}_{\text{pf}}\right). \tag{8}$$

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[3]The metric projection P_D is defined by $P_D(x) \in D$ and $\|x - P_D(x)\| = \inf_{y \in D} \|x - y\|$ ($x \in \mathbb{R}^S$), where $\|\cdot\|$ is the norm of $\mathbb{R}^S$.

[4]T is said to be nonexpansive if $\|T(x) - T(y)\| \leq \|x - y\|$ ($x, y \in \mathbb{R}^S$).

[5]Q is said to be quasi-nonexpansive if $\|Q(x) - y\| \leq \|x - y\|$ ($x \in \mathbb{R}^S, y \in \text{Fix}(Q) := \{y \in \mathbb{R}^S \colon Q(y) = y\}$). R is said to be quasi-firmly nonexpansive if there exists a quasi-nonexpansive mapping Q such that $R(x) = (1/2)(x + Q(x))$ ($x \in \mathbb{R}^S$).

[6]Q is fixed-point closed if $x \in \text{Fix}(Q)$ whenever $(x_n)_{n \in \mathbb{N}}$ ($\subset \mathbb{R}^S$) converges to x ($\in \mathbb{R}^S$) and $\lim_{n \to \infty} \|x_n - Q(x_n)\| = 0$.

[7]We can prove that $\langle x - y, \nabla f(x) - \nabla f(y) \rangle \geq (\min_{s \in \mathcal{S}} w_s / (\max_{s \in \mathcal{S}} b_s)^2) \|x - y\|^2$ and $\|\nabla f(x) - \nabla f(y)\| \leq (\max_{s \in \mathcal{S}} w_s / \min_{s \in \mathcal{S}} a_s) \|x - y\|$ for all $x, y \in \Pi_{s \in \mathcal{S}} [a_s, b_s]$, where $0 < a_s \leq b_s < \infty$ ($s \in \mathcal{S}$) and $f := -\mathcal{U}_{\text{pf}}$. See Subsection 2.2 for the definitions of strongly monotone and Lipschitz continuous operators.

1.4. Main problem and our objective

This paper discusses a more general variational inequality that includes Problem (8).

Problem 1.2. *Assume that*

(A1) $Q\colon \mathbb{R}^S \to \mathbb{R}^S$ *is quasi-firmly nonexpansive and fixed-point closed;*
(A2) $T\colon \mathbb{R}^S \to \mathbb{R}^S$ *is nonexpansive with* $X := \mathrm{Fix}(T) \cap \mathrm{Fix}(Q) \neq \emptyset$*;*
(A3) $f\colon \mathbb{R}^S \to \mathbb{R}$ *is continuously differentiable, and* $\nabla f\colon \mathbb{R}^S \to \mathbb{R}^S$ *is strongly monotone with a constant* $c > 0$ *and Lipschitz continuous with a constant* $L > 0$*.*

Our objective is to

$$\text{find } x^\star \in \mathrm{VI}(\mathrm{Fix}(T) \cap \mathrm{Fix}(Q), \nabla f) := \left\{ x^\star \in X\colon \langle x - x^\star, \nabla f(x^\star)\rangle \geq 0 \ (x \in X)\right\}.$$

The main objective of the paper is to devise an iterative algorithm, based on iterative techniques for convex optimization over fixed point sets [8, 17, 20, 21, 39, 41] for solving Problem 1.2 and its convergence analysis.

1.5. Related work, the proposed algorithm, and the contributions of this paper

Let us consider the case where $\mathrm{Fix}(T) \cap \mathrm{Fix}(Q)$ in Problem 1.2 is simple in the sense that the metric projection onto $\mathrm{Fix}(T) \cap \mathrm{Fix}(Q)$ can be easily calculated.[8] The projected gradient methods [6, 14] can be applied to Problem 1.2 in this case. Generally, the projection onto $\mathrm{Fix}(T) \cap \mathrm{Fix}(Q)$ cannot be easily calculated. For example, the projection onto C in (5) cannot be easily calculated because $C := \mathbb{R}_+^S \cap \bigcap_{l \in \mathcal{L}} C_l$ is polyhedral. Meanwhile, the nonexpansive mapping $T_{\mathrm{proj}} := P_{\mathbb{R}_+^S} \prod_{l \in \mathcal{L}} P_{C_l}$ satisfying $\mathrm{Fix}(T_{\mathrm{proj}}) = C$ (see also (4) and (5)) can be computed because $\mathbb{R}_+^S$ and C_ls are half-spaces. From this viewpoint, a number of iterative algorithms that use nonexpansive mappings have been developed for solving Problem 1.2. These are referred to here as *fixed point optimization algorithms*.

[8]The metric projection onto a closed ball, a closed cone, or a half-space can be easily calculated and satisfies the nonexpansivity condition. The metric projection onto a half-space, $H := \{x \in \mathbb{R}^S\colon \langle a, x\rangle \leq b\}$, where $a\ (\neq 0) \in \mathbb{R}^S$ and $b \in \mathbb{R}$, is expressed as $P_H(x) := x - [\max\{0, \langle a, x\rangle - b\}/\|a\|^2]a\ (x \in \mathbb{R}^S)$ [1, p. 406], [4, Subchapter 28.3]

There are fixed point optimization algorithms [8, 20, 39, 41] for solving Problem 1.2 when Q is the identity mapping I, T is nonexpansive, and ∇f is strongly monotone and Lipschitz continuous. Combettes [8] presented a block-iterative surrogate constraint splitting method without using diminishing sequences and applied it to signal recovery. A hybrid steepest descent method (HSDM) [39, 41] was applied to beamforming [35]. Iiduka and Uchida [19] applied fixed point optimization algorithms [20, 41] to bandwidth allocation problems for concave utility functions in which the constraints about the preferable transmission rate fall in the infeasible region. Iiduka [17] presented distributed fixed point optimization algorithms for Problem 1.2 when $Q := I$, T is 1/2-averaged nonexpansive, and ∇f is strictly monotone. Fixed point optimization algorithms [15, 16, 19] were presented for solving Problem 1.2 when $Q := I$, T is 1/2-averaged nonexpansive, and ∇f is continuous. The algorithm [16] was applied to bandwidth allocation problems for nonconcave utility functions in which the constraints about the preferable transmission rate fall in the infeasible region. An application of the algorithm [15] to power control was discussed in [15]. Yamada and Ogura [40] applied HSDM to Problem 1.2 when Q is quasi-nonexpansive on $\mathbb{R}^S$ and quasi-shrinking on a certain set, $T := I$, and ∇f is strongly monotone and Lipschitz continuous, and proved that HSDM converges to the solution to Problem 1.2 in this case [40, Theorem 4].

In this paper, we devise a fixed point optimization algorithm for solving the original Problem 1.2, which the existing algorithms described in the above paragraph cannot solve.

The contribution of this paper is that it is the first study to tackle variational inequality problems over the fixed point sets of a quasi-nonexpansive mapping and a nonexpansive mapping and it proposes a fixed point optimization algorithm for these variational inequality problems. We can apply the algorithm to utility-based bandwidth allocation problems with concave utility functions, and it can determine an optimal bandwidth allocation. The problem of minimizing a function f with (A3) over the fixed point set of a nonexpansive mapping T includes other practical network resource allocation problems, such as power allocation [34], channel allocation [22], and storage allocation [18, 26]. When the operational constraint set can be expressed as the fixed point set of a certain quasi-nonexpansive mapping (see (2)), practical network resource allocation problems with operational constraints can be formulated as Problem 1.2. Therefore, our algorithm can be applied to not only bandwidth allocation but also other network resource allocations.

The organization of the paper is as follows. Section 2 provides the necessary mathematical preliminaries. Section 3 presents a fixed point optimization algorithm (Algorithm 3.1) for solving Problem 1.2. It proves that Algorithm 3.1 converges to the unique solution to Problem 1.2 under certain assumptions (Theorem 3.1). Section 4 applies the algorithm to concrete utility-based bandwidth allocation problem and provides numerical examples. Section 5 concludes the paper by summarizing the key points.

2. Mathematical Preliminaries

This section gives necessary mathematical preliminaries. Let $\mathbb{N}$ be the set of all positive integers and zero, i.e., $\mathbb{N} := \{0, 1, 2, \ldots\}$, let $\mathbb{R}^S$ be an S-dimensional Euclidean space with inner product, $\langle \cdot, \cdot \rangle$, and its induced norm, $\| \cdot \|$, and let $\mathbb{R}^S_+ := \{x := (x_1, x_2, \ldots, x_S)^T \in \mathbb{R}^S \colon x_s \geq 0 \ (s = 1, 2, \ldots, S)\}$. The *fixed point set* of a mapping, $T \colon \mathbb{R}^S \to \mathbb{R}^S$, is denoted by $\mathrm{Fix}(T) := \{x \in \mathbb{R}^S \colon T(x) = x\}$. Let I be the identity mapping on $\mathbb{R}^S$.

2.1. Nonexpansivity and quasi nonexpansivity

A mapping, $T \colon \mathbb{R}^S \to \mathbb{R}^S$, is said to be *nonexpansive* [1], [12, Chapter 3], [13, Chapter 1] if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in \mathbb{R}^S$. Obviously, any nonexpansive mapping satisfies continuity. The fixed point set of any nonexpansive mapping satisfies closedness and convexity [13, Proposition 5.3]. A well-known example of a nonexpansive mapping is the metric projection, $P_D \colon \mathbb{R}^S \to \mathbb{R}^S$, onto a nonempty, closed convex set, D ($\subset \mathbb{R}^S$), defined by $P_D(x) \in D$ and $\|x - P_D(x)\| = \inf_{y \in D} \|x - y\|$.

A mapping, $Q \colon \mathbb{R}^S \to \mathbb{R}^S$, is said to be *quasi-nonexpansive* [4, Definition 4.1 (iii)] if $\|Q(x) - y\| \leq \|x - y\|$ for all $x \in \mathbb{R}^S$ and for all $y \in \mathrm{Fix}(Q)$. When a quasi-nonexpansive mapping has one fixed point, its fixed point set is closed and convex [3, Proposition 2.6], [40, Proposition 1 (a)]. $R \colon \mathbb{R}^S \to \mathbb{R}^S$ is called an α-*averaged quasi-nonexpansive* mapping if $\alpha \in (0, 1]$ and a quasi-nonexpansive mapping, $Q \colon \mathbb{R}^S \to \mathbb{R}^S$, exist such that $R = \alpha I + (1 - \alpha) Q$. In particular, a 1/2-averaged quasi-nonexpansive mapping is called a *quasi-firmly nonexpansive* mapping. We shall provide an important example of a quasi-firmly nonexpansive mapping. Let $f_0 \colon \mathbb{R}^S \to \mathbb{R}$ be a convex function with $\mathrm{lev}_{\leq 0} f_0 := \{x \in \mathbb{R}^S \colon f_0(x) \leq 0\} \neq \emptyset$. Then, the *subdifferential* [32, Section 23] of f_0 at $x \in \mathbb{R}^S$, denoted by $\partial f_0(x) := \{z \in \mathbb{R}^S \colon f_0(y) \geq f_0(x) + \langle y - x, z \rangle \ (y \in \mathbb{R}^S)\}$, has a point and the subgradient of f_0 at x is denoted by $f_0'(x) \in \partial f_0(x)$. The *subgradient projection relative to* f_0 [3, Proposition 2.3], [38, Lemma 2.8 in Subchapter 2.4.3], $Q_{\mathrm{sp}} \colon \mathbb{R}^S \to \mathbb{R}^S$, defined for all $x \in \mathbb{R}^S$ by

$$Q_{\mathrm{sp}}(x) := \begin{cases} x - \dfrac{f_0(x)}{\|f_0'(x)\|^2} f_0'(x) & \text{if } f_0(x) > 0, \\ x & \text{otherwise} \end{cases}$$

is quasi-firmly nonexpansive (i.e., $2Q_{\mathrm{sp}} - I$ is quasi-nonexpansive) and satisfies $\mathrm{Fix}(Q_{\mathrm{sp}}) = \mathrm{Fix}(2Q_{\mathrm{sp}} - I) = \mathrm{lev}_{\leq 0} f_0$. Moreover, we have the following:

Proposition 2.1.

(i) [2, Lemma 3.1] Q_{sp} *is fixed-point closed, i.e.,* $x \in \mathrm{Fix}(Q_{\mathrm{sp}})$ *whenever* $(x_n)_{n\in\mathbb{N}}$ $(\subset \mathbb{R}^S)$ *converges to* x $(\in \mathbb{R}^S)$ *and* $\lim_{n\to\infty} \|x_n - Q_{\mathrm{sp}}(x_n)\| = 0$*;*
(ii) $2Q_{\mathrm{sp}} - I$ *is fixed-point closed.*

It is obvious from $\mathrm{Fix}(Q_{\mathrm{sp}}) = \mathrm{Fix}(2Q_{\mathrm{sp}} - I)$ and Proposition 2.1(i) that Proposition 2.1(ii) holds. Any strictly pseudo-contractive mapping[9] satisfies the fixed-point closedness condition [28, Proposition 2.1(ii)]. This means that any nonexpansive mapping also satisfies the fixed-point closedness condition.

The following lemma indicates the properties of quasi-firmly nonexpansive mappings.

Lemma 2.1. [27, Remark 2.1] *Suppose that* Q *is quasi-firmly nonexpansive with* $\mathrm{Fix}(Q) \neq \emptyset$ *and* $\alpha \in (0, 1]$*, and define* $Q_\alpha := \alpha I + (1 - \alpha)Q$*. Then, the following hold:*

(i) $\mathrm{Fix}(Q) = \mathrm{Fix}(Q_\alpha)$*;*
(ii) Q_α *is quasi-nonexpansive;*
(iii) $\langle x - Q_\alpha(x), x - y\rangle \geq (1 - \alpha)\|x - Q(x)\|^2$ $(x \in \mathbb{R}^S, y \in \mathrm{Fix}(Q))$.

2.2. Monotone variational inequality

An operator, $A \colon \mathbb{R}^S \to \mathbb{R}^S$, is said to be *monotone* [43, Definition 25.2 (i)] if $\langle x - y, A(x) - A(y)\rangle \geq 0$ for all $x, y \in \mathbb{R}^S$. $A \colon \mathbb{R}^S \to \mathbb{R}^S$ is called a strongly monotone operator with $c > 0$ (c-strongly monotone operator) [43, Definition 25.2 (iii)] if $\langle x - y, A(x) - A(y)\rangle \geq c\|x - y\|^2$ for all $x, y \in \mathbb{R}^S$. $A \colon \mathbb{R}^S \to \mathbb{R}^S$ is

[9] T is said to be strictly pseudo-contractive if there exists $\alpha \in [0, 1)$ such that $\|T(x) - T(y)\|^2 - \alpha\|(x - T(x)) - (y - T(y))\|^2 \leq \|x - y\|^2$ $(x, y \in \mathbb{R}^S)$. The class of strictly pseudo-contractive mappings includes the class of nonexpansive mappings.

called a Lipschitz continuous operator with $L > 0$ (L-Lipschitz continuous) if $\|A(x) - A(y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^S$. Define $f\colon \mathbb{R}^S \to \mathbb{R}$ for all $x \in \mathbb{R}^S$ by $f(x) := (1/2)\langle x, Ax\rangle + \langle b, x\rangle$, where $A \in \mathbb{R}^{S\times S}$ is positive definite and $b \in \mathbb{R}^S$. Then, $\nabla f(x) = Ax + b$ is $\lambda_{\min}$-strongly monotone and $\lambda_{\max}$-Lipschitz continuous, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum eigenvalue and maximum eigenvalue of A, respectively [39, Lemma 2.9]. The gradient of the weighted proportionally fair function also satisfies the strong monotonicity and Lipschitz continuity conditions (See Footnote 7).

The following lemma is used to prove Theorem 3.1:

Lemma 2.2. [39, Lemma 3.1] *Suppose that $A\colon \mathbb{R}^S \to \mathbb{R}^S$ is c-strongly monotone and L-Lipschitz continuous and $\mu \in (0, 2c/L^2)$. For $\alpha \in [0, 1]$, define $T_\alpha\colon \mathbb{R}^S \to \mathbb{R}^S$ by $T_\alpha(x) := x - \mu\alpha A(x)$ for all $x \in \mathbb{R}^S$. Then, for all $x, y \in \mathbb{R}^S$,*

$$\|T_\alpha(x) - T_\alpha(y)\| \leq (1 - \tau\alpha)\|x - y\|$$

holds, where $\tau := 1 - \sqrt{1 - \mu(2c - \mu L^2)} \in (0, 1]$.

The *variational inequality* problem [9, Chapter II], [24, Chapter I] for a monotone operator, $A\colon \mathbb{R}^S \to \mathbb{R}^S$, over a nonempty, closed convex set, $D \subset \mathbb{R}^S$, is to find a point in

$$\mathrm{VI}(D, A) := \{x^\star \in D\colon \langle y - x^\star, A(x^\star)\rangle \geq 0 \ (y \in D)\}.$$

Some properties of the solution set of the monotone variational inequality are as follows:

Proposition 2.2. *Suppose that D $(\subset \mathbb{R}^S)$ is nonempty, closed, and convex, $A\colon \mathbb{R}^S \to \mathbb{R}^S$ is continuous, and $f\colon \mathbb{R}^S \to \mathbb{R}$ is convex and differentiable. Then,*

- (a) [9, Chapter 2, Proposition 2.1 (2.1) and (2.2)] $\mathrm{VI}(D, \nabla f) = \operatorname{argmin}_{x\in D} f(x) := \{x^\star \in D\colon f(x^\star) = \min_{x\in D} f(x)\}$.
- (b) [10, Corollary 2.2.5] $\mathrm{VI}(D, A) \neq \emptyset$ *when D is compact.*
- (c) [38, Theorem 2.31] $\mathrm{VI}(D, A)$ *consists of one point, if A is strongly monotone and Lipschitz continuous.*

The closedness and convexity [13, Proposition 5.3], [3, Proposition 2.6], [40, Proposition 1(a)] of $\mathrm{Fix}(T) \cap \mathrm{Fix}(Q)$ $(\neq \emptyset)$ and (A3) guarantee the existence and uniqueness of the solution to Problem 1.2 (Proposition 2.2(c)).

We need the following useful lemma to prove the main theorem.

Lemma 2.3. [27, Lemma 2.1] *Let* $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ *and suppose that* $(\Gamma_{n_j})_{j\in\mathbb{N}}$ $(\subset (\Gamma_n)_{n\in\mathbb{N}})$ *exists such that* $\Gamma_{n_j} < \Gamma_{n_j+1}$ *for all* $j \in \mathbb{N}$. *Define* $(\tau(n))_{n\geq n_0}$ $(\subset \mathbb{N})$ *by* $\tau(n) := \max\{k \leq n \colon \Gamma_k < \Gamma_{k+1}\}$ *for some* $n_0 \in \mathbb{N}$. *Then,* $(\tau(n))_{n\geq n_0}$ *is increasing and* $\lim_{n\to\infty} \tau(n) = \infty$. *Moreover,* $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$ *for all* $n \geq n_0$.

3. Fixed Point Optimization Algorithm for Solving Problem 1.2

This section presents the following algorithm:

Algorithm 3.1 (Fixed point optimization algorithm)**.**
Step 0. Take $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}} \subset (0,\infty)$, *and* $\mu, \alpha > 0$, *and define* $Q_\alpha := \alpha I + (1-\alpha)Q$. *Choose* $x_0 \in \mathbb{R}^S$ *arbitrarily, and let* $n := 0$.
Step 1. Given $x_n \in \mathbb{R}^S$, *compute* $x_{n+1} \in \mathbb{R}^S$ *as*

$$\begin{cases} y_n := Q_\alpha\left(x_n\right) - \mu\alpha_n \nabla f\left(Q_\alpha\left(x_n\right)\right), \\ x_{n+1} := \beta_n x_n + (1-\beta_n) T\left(y_n\right). \end{cases}$$

Update $n := n+1$ *and go to Step 1.*

The following theorem constitutes the convergence analysis of Algorithm 3.1.

Theorem 3.1. *Suppose that* $\mu \in (0, 2c/L^2)$, *and* $(\alpha_n)_{n\in\mathbb{N}} \subset (0,1]$ *and* $(\beta_n)_{n\in\mathbb{N}} \subset (0,1)$ *satisfy*[10] (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Then,* $(x_n)_{n\in\mathbb{N}}$ *in Algorithm 3.1 converges to the unique solution to Problem 1.2.*

Let us compare HSDM [40] with Algorithm 3.1. The sequence, $(x_n)_{n\in\mathbb{N}}$, generated by HSDM is

$$x_{n+1} := Q(x_n) - \alpha_n \nabla f(Q(x_n)) \ (n \in \mathbb{N}),$$

where $(\alpha_n)_{n\in\mathbb{N}} \subset (0,\infty)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Theorem 4 in [40] guarantees that HSDM converges to the unique solution to Problem 1.2 when $T := I$ and Q is quasi-nonexpansive if $\bar{x} \in \mathrm{Fix}(Q)$ and

[10] Examples satisfying Conditions (i)-(iii) are $\alpha_n := 1/(n+1)^a$ and $\beta_n := b$ $(n \in \mathbb{N})$, where $a \in (0,1]$ and $b \in (0,1)$.

$\bar{\mu} \in (0, 2c/L^2)$ exist such that Q is quasi-shrinking[11] on $B_{\bar{x}}(\rho(x_0)) := \{x \in \mathbb{R}^S : \|x-\bar{x}\| \leq \rho(x_0)\}$, where $\rho(x_0) := \max\{\|\bar{\mu}\nabla f(\bar{x})\|/\tau, \|x_0-\bar{x}\|, \max_{\alpha_n > \bar{\mu}} \|x_n - \bar{x}\|\}$ and $\tau := 1 - \sqrt{1-\bar{\mu}(2c - \bar{\mu}L^2)} \in (0,1]$. It would be difficult to check for the existence of $B_{\bar{x}}(\rho(x_0))$ on which Q is quasi-shrinking before executing HSDM. Hence, HSDM will not converge even in $\mathrm{Fix}(Q)$ when Q is only quasi-nonexpansive.

Meanwhile, Theorem 3.1 in this paper guarantees that Algorithm 3.1 converges to the unique solution to Problem 1.2 when T is nonexpansive and Q is quasi-firmly nonexpansive and fixed-point closed. Therefore, we can conclude that Algorithm 3.1 does not require us to check whether complicated assumptions, such as the existence of $B_{\bar{x}}(\rho(x_0))$, are satisfied or not in advance, and it can solve Problem 1.2, which HSDM cannot solve.

3.1. Proof of Theorem 3.1

We shall prove Theorem 3.1 by referring to the proof of [27, Theorem 3.1]. We first prove the following:

Lemma 3.1. *Suppose that the assumptions of Theorem 3.1 hold. Then,* $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, *and* $(\nabla f(Q_\alpha(x_n)))_{n\in\mathbb{N}}$ *are bounded.*

Proof. Let $x \in \mathrm{Fix}(T) \cap \mathrm{Fix}(Q)$ be arbitrarily fixed. Lemma 2.2, the quasi nonexpansivity of Q_α, and $\mathrm{Fix}(Q) = \mathrm{Fix}(Q_\alpha)$ (Lemma 2.1(i), (ii)) mean that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|y_n - x\| &= \|Q_\alpha(x_n) - \mu\alpha_n \nabla f(Q_\alpha(x_n)) - x\| \\
&\leq \|(Q_\alpha(x_n) - \mu\alpha_n \nabla f(Q_\alpha(x_n))) - (x - \mu\alpha_n \nabla f(x))\| \\
&\quad + \mu\alpha_n \|\nabla f(x)\| \\
&\leq (1-\tau\alpha_n)\|Q_\alpha(x_n) - x\| + \mu\alpha_n \|\nabla f(x)\| \\
&\leq (1-\tau\alpha_n)\|x_n - x\| + \mu\alpha_n \|\nabla f(x)\|,
\end{aligned}$$

where $\tau := 1 - \sqrt{1-\mu(2c-\mu L^2)} \in (0,1]$. Accordingly, for all $n \in \mathbb{N}$, we have

$$\|y_n - x\| \leq (1-\tau\alpha_n)\|x_n - x\| + \frac{\mu\|\nabla f(x)\|}{\tau}\tau\alpha_n. \tag{9}$$

[11]See [40] for the definition of a quasi-shrinking mapping.

The triangle inequality, nonexpansivity of T, and (9) ensure that

$$\begin{aligned}\|x_{n+1}-x\| &= \|\beta_n x_n + (1-\beta_n)T(y_n) - x\| \\ &\leq \beta_n\|x_n - x\| + (1-\beta_n)\,\|T(y_n)-x\| \\ &\leq \beta_n\|x_n - x\| + (1-\beta_n)\,\|y_n - x\| \\ &\leq \beta_n\|x_n - x\| + (1-\beta_n)\left\{(1-\tau\alpha_n)\|x_n - x\| + \frac{\mu\|\nabla f(x)\|}{\tau}\tau\alpha_n\right\} \\ &= (1-\tau(1-\beta_n)\alpha_n)\,\|x_n - x\| + \frac{\mu\|\nabla f(x)\|}{\tau}\tau(1-\beta_n)\alpha_n.\end{aligned}$$

Induction shows that, for all $n \in \mathbb{N}$,

$$\|x_n - x\| \leq \max\left\{\|x_0 - x\|, \frac{\mu\|\nabla f(x)\|}{\tau}\right\};$$

that is, $(x_n)_{n\in\mathbb{N}}$ is bounded. The Lipschitz continuity of ∇f, the quasi nonexpansivity of Q_α, and the boundedness of $(x_n)_{n\in\mathbb{N}}$ ensure that $(\nabla f(Q_\alpha(x_n)))_{n\in\mathbb{N}}$ is bounded. Inequality (9) and the boundedness of $(x_n)_{n\in\mathbb{N}}$ imply that $(y_n)_{n\in\mathbb{N}}$ is bounded. This completes the proof. □

Now let us prove the following lemma.

Lemma 3.2. *Let* $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^S$ *be a bounded sequence with* $\lim_{n\to\infty}\|z_n - Q(z_n)\| = 0$ *and* $\lim_{n\to\infty}\|z_n - T(z_n)\| = 0$. *Then,* $\liminf_{n\to\infty}\langle z_n - x^\star, \nabla f(x^\star)\rangle \geq 0$, *where* $x^\star \in \mathrm{Fix}(T)\cap\mathrm{Fix}(Q)$ *stands for the solution to Problem 1.2.*

Proof. From the limit inferior of $(\langle z_n - x^\star, \nabla f(x^\star)\rangle)_{n\in\mathbb{N}}$, there exists a subsequence (z_{n_i}) of $(z_n)_{n\in\mathbb{N}}$ such that

$$\liminf_{n\to\infty}\langle z_n - x^\star, \nabla f(x^\star)\rangle = \lim_{i\to\infty}\langle z_{n_i} - x^\star, \nabla f(x^\star)\rangle. \tag{10}$$

Since $(z_{n_i})_{i\in\mathbb{N}}$ is bounded, there exists $(z_{n_{i_j}})_{j\in\mathbb{N}}$ ($\subset (z_{n_i})_{i\in\mathbb{N}}$) converging to $\bar{x} \in \mathbb{R}^S$. We may assume without loss of generality that $(z_{n_i})_{i\in\mathbb{N}}$ converges to $\bar{x} \in \mathbb{R}^S$. The demiclosedness principle [4, Theorem 4.17, Corollary 4.18] of T guarantees $\bar{x} \in \mathrm{Fix}(T)$. Since Q is quasi-nonexpansive and fixed-point closed, we also have $\bar{x} \in \mathrm{Fix}(Q)$, and hence, $\bar{x} \in \mathrm{Fix}(T)\cap\mathrm{Fix}(Q)$. Therefore, (10) leads one to deduce that

$$\liminf_{n\to\infty}\langle z_n - x^\star, \nabla f(x^\star)\rangle = \langle \bar{x} - x^\star, \nabla f(x^\star)\rangle \geq 0, \tag{11}$$

where the first inequality comes from $x^\star \in \mathrm{VI}(\mathrm{Fix}(T)\cap\mathrm{Fix}(Q), \nabla f)$. This completes the proof. □

Next, we prove the following lemma.

Lemma 3.3. *For all* $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - x^\star\|^2 \leq \|x_n - x^\star\|^2 &- 2\alpha(1-\alpha)(1-\beta_n)\|Q(x_n) - x_n\|^2 \\ &+ 2\mu^2\alpha_n^2(1-\beta_n)\left\|\nabla f\left(Q_\alpha(x_n)\right)\right\|^2 - \beta_n(1-\beta_n)\|x_n - T(y_n)\|^2 \\ &- 2\mu\alpha_n(1-\beta_n)\left\langle x_n - x^\star, \nabla f\left(Q_\alpha(x_n)\right)\right\rangle.\end{aligned}$$

Proof. From $-2\langle x, y\rangle = \|x-y\|^2 - \|x\|^2 - \|y\|^2$ $(x, y \in \mathbb{R}^S)$, we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}&2\left\langle y_n - x_n + \mu\alpha_n \nabla f(Q_\alpha(x_n)), x_n - x^\star\right\rangle \\ &= -2\left\langle x_n - y_n, x_n - x^\star\right\rangle + 2\mu\alpha_n\left\langle \nabla f(Q_\alpha(x_n)), x_n - x^\star\right\rangle \\ &= \|y_n - x^\star\|^2 - \|x_n - x^\star\|^2 - \|x_n - y_n\|^2 + 2\mu\alpha_n\left\langle \nabla f(Q_\alpha(x_n)), x_n - x^\star\right\rangle.\end{aligned}$$

Meanwhile, the quasi-firm nonexpansivity of Q and Lemma 2.1(iii) guarantee that, for all $n \in \mathbb{N}$,

$$2\left\langle x_n - Q_\alpha(x_n), x_n - x^\star\right\rangle \geq 2(1-\alpha)\|x_n - Q(x_n)\|^2.$$

Since the definition of y_n means that $y_n - x_n + \mu\alpha_n\nabla f(Q_\alpha(x_n)) = Q_\alpha(x_n) - x_n$ $(n \in \mathbb{N})$, we find that, for all $n \in \mathbb{N}$,

$$\begin{aligned}-2(1-\alpha)\|x_n - Q(x_n)\|^2 \geq \|y_n - x^\star\|^2 &- \|x_n - x^\star\|^2 - \|x_n - y_n\|^2 \\ &+ 2\mu\alpha_n\left\langle \nabla f(Q_\alpha(x_n)), x_n - x^\star\right\rangle,\end{aligned}$$

which means that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|y_n - x^\star\|^2 \leq \|x_n - x^\star\|^2 &+ \|x_n - y_n\|^2 - 2(1-\alpha)\|x_n - Q(x_n)\|^2 \\ &- 2\mu\alpha_n\left\langle \nabla f(Q_\alpha(x_n)), x_n - x^\star\right\rangle.\end{aligned} \tag{12}$$

Moreover, from $\|x-y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ $(x, y \in \mathbb{R}^S)$,

$$\begin{aligned}\|x_n - y_n\|^2 &= \left\|(x_n - Q_\alpha(x_n)) + \mu\alpha_n\nabla f\left(Q_\alpha(x_n)\right)\right\|^2 \\ &\leq 2\left\|x_n - Q_\alpha(x_n)\right\|^2 + 2\mu^2\alpha_n^2\left\|\nabla f\left(Q_\alpha(x_n)\right)\right\|^2 \\ &= 2(1-\alpha)^2\left\|x_n - Q(x_n)\right\|^2 + 2\mu^2\alpha_n^2\left\|\nabla f\left(Q_\alpha(x_n)\right)\right\|^2.\end{aligned} \tag{13}$$

Hence, (12) and (13) imply that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|y_n - x^\star\|^2 \leq \|x_n - x^\star\|^2 &- 2\alpha(1-\alpha)\|x_n - Q(x_n)\|^2 \\ &+ 2\mu^2\alpha_n^2\left\|\nabla f\left(Q_\alpha(x_n)\right)\right\|^2 - 2\mu\alpha_n\left\langle \nabla f(Q_\alpha(x_n)), x_n - x^\star\right\rangle.\end{aligned} \tag{14}$$

Since $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ $(x,y \in \mathbb{R}^S, t \in [0,1])$, the nonexpansivity of T means that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\|x_{n+1}-x^\star\|^2 \\
&= \|\beta_n(x_n - x^\star) + (1-\beta_n)(T(y_n)-x^\star)\|^2 \\
&= \beta_n\|x_n - x^\star\|^2 + (1-\beta_n)\|T(y_n)-x^\star\|^2 - \beta_n(1-\beta_n)\|x_n - T(y_n)\|^2 \\
&\leq \beta_n\|x_n - x^\star\|^2 + (1-\beta_n)\|y_n-x^\star\|^2 - \beta_n(1-\beta_n)\|x_n - T(y_n)\|^2.
\end{aligned}$$

We find from (14) that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|x_{n+1}-x^\star\|^2 \leq \|x_n - x^\star\|^2 &- 2\alpha(1-\alpha)(1-\beta_n)\|Q(x_n)-x_n\|^2 \\
&+ 2\mu^2\alpha_n^2(1-\beta_n)\left\|\nabla f\left(Q_\alpha(x_n)\right)\right\|^2 - \beta_n(1-\beta_n)\|x_n - T(y_n)\|^2 \\
&- 2\mu\alpha_n(1-\beta_n)\left\langle x_n - x^\star, \nabla f\left(Q_\alpha(x_n)\right)\right\rangle.
\end{aligned}$$

This completes the proof. □

We are now in a position to prove Theorem 3.1.

Proof. Let us consider the case where $n_0 \in \mathbb{N}$ exists such that $\|x_{n+1}-x^\star\| \leq \|x_n - x^\star\|$ for all $n \geq n_0$. Then, $\lim_{n\to\infty}\|x_n - x^\star\|$ exists. Lemma 3.3 guarantees that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&2\alpha(1-\alpha)(1-\beta_n)\|x_n - Q(x_n)\|^2 \leq \|x_n - x^\star\|^2 - \|x_{n+1}-x^\star\|^2 + M\alpha_n, \\
&\beta_n(1-\beta_n)\|x_n - T(y_n)\|^2 \leq \|x_n - x^\star\|^2 - \|x_{n+1}-x^\star\|^2 + M\alpha_n,
\end{aligned}$$

where

$$M_n := 2\mu(1-\beta_n)\left\{\mu\alpha_n\|\nabla f(Q_\alpha(x_n))\|^2 - \langle x_n - x^\star, \nabla f(Q_\alpha(x_n))\rangle\right\} \ (n \in \mathbb{N})$$

and $M := \sup_{n\in\mathbb{N}} M_n$ ($M < \infty$ holds from Lemma 3.1). Accordingly, (i) and (iii) in Theorem 3.1 and the existence of $\lim_{n\to\infty}\|x_n - x^\star\|$ ensure that

$$\lim_{n\to\infty}\|x_n - Q(x_n)\| = 0 \text{ and } \lim_{n\to\infty}\|x_n - T(y_n)\| = 0. \tag{15}$$

Moreover, from (13), (15), and (i) in Theorem 3.1, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

Since the triangle inequality and the nonexpansivity of T ensure that $\|x_n - T(x_n)\| \leq \|x_n - T(y_n)\| + \|T(y_n) - T(x_n)\| \leq \|x_n - T(y_n)\| + \|y_n - x_n\|$ ($n \in \mathbb{N}$), we find

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = 0. \tag{16}$$

Accordingly, Lemmas 3.1 and 3.2, (15), and (16) guarantee that

$$\liminf_{n\to\infty} \langle x_n - x^\star, \nabla f(x^\star)\rangle \geq 0. \tag{17}$$

Lemma 3.3 implies that, for all $N \in \mathbb{N}$,

$$\sum_{n=0}^{N} \alpha_n (-M_n) \leq \|x_0 - x^\star\|^2 - \|x_{N+1} - x^\star\|^2 \leq \|x_0 - x^\star\|^2 < \infty,$$

and hence,

$$\sum_{n=0}^{\infty} \alpha_n (-M_n) < \infty.$$

Assume that $\liminf_{n\to\infty}(-M_n) > 0$. Then, since there exist $n_1 \in \mathbb{N}$ and $\delta > 0$ such that $-M_n \geq \delta$ for all $n \geq n_1$, we have that $\delta\alpha_n \leq \alpha_n(-M_n)$ ($n \geq n_1$), which, together with (ii) in Theorem 3.1, means that

$$\infty = \delta \sum_{n=n_1}^{\infty} \alpha_n \leq \sum_{n=n_1}^{\infty} \alpha_n(-M_n) < \infty.$$

This is a contradiction. Hence, we find that

$$\liminf_{n\to\infty} \left\{-\mu\alpha_n \|\nabla f(Q_\alpha(x_n))\|^2 + \langle x_n - x^\star, \nabla f(Q_\alpha(x_n))\rangle\right\} \leq 0,$$

which, together with (i) in Theorem 3.1, implies that

$$\liminf_{n\to\infty} \langle x_n - x^\star, \nabla f(Q_\alpha(x_n))\rangle \leq 0. \tag{18}$$

The Cauchy-Schwarz inequality and the Lipschitz continuous of ∇f (Assumption (A3)) mean that, for all $n \in \mathbb{N}$,

$$\begin{aligned}\langle x_n - x^\star, \nabla f(x_n)\rangle &= \langle x_n - x^\star, \nabla f(x_n) - \nabla f(Q_\alpha(x_n))\rangle \\ &\quad + \langle x_n - x^\star, \nabla f(Q_\alpha(x_n))\rangle \\ &\leq L\|x_n - x^\star\|\|x_n - Q_\alpha(x_n)\| + \langle x_n - x^\star, \nabla f(Q_\alpha(x_n))\rangle.\end{aligned}$$

Since (15) implies $\lim_{n\to\infty}\|x_n - Q_\alpha(x_n)\| = (1-\alpha)\lim_{n\to\infty}\|x_n - Q(x_n)\| = 0$, we find from (18) that

$$\liminf_{n\to\infty}\langle x_n - x^\star, \nabla f(x_n)\rangle \leq 0. \tag{19}$$

On the other hand, the strong monotonicity of ∇f (Assumption (A3)) ensures the existence of $c > 0$ such that, for all $n \in \mathbb{N}$,

$$c\|x_n - x^\star\|^2 \leq \langle x_n - x^\star, \nabla f(x_n)\rangle - \langle x_n - x^\star, \nabla f(x^\star)\rangle. \tag{20}$$

Therefore, from (17), (19), and (20),

$$\begin{aligned} c\lim_{n\to\infty}\|x_n - x^\star\|^2 &\leq c\lim_{n\to\infty}\|x_n - x^\star\|^2 + \liminf_{n\to\infty}\langle x_n - x^\star, \nabla f(x^\star)\rangle \\ &= \liminf_{n\to\infty}\left\{c\|x_n - x^\star\|^2 + \langle x_n - x^\star, \nabla f(x^\star)\rangle\right\} \\ &\leq \liminf_{n\to\infty}\langle x_n - x^\star, \nabla f(x_n)\rangle \\ &\leq 0, \end{aligned}$$

which implies that $(x_n)_{n\in\mathbb{N}}$ converges to $x^\star$.

Next, we define $\Gamma_n := \|x_n - x^\star\|^2$ ($n \in \mathbb{N}$) and consider the case where $(\Gamma_{n_j})_{j\in\mathbb{N}}$ ($\subset (\Gamma_n)_{n\in\mathbb{N}}$) exists such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \in \mathbb{N}$. Then, Lemma 2.3 implies that $n_2 \in \mathbb{N}$ exists such that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ ($n \geq n_2$), where $\tau(n)$ is as in Lemma 2.3. A discussion in the same manner as in the proof of (15) and (16) leads us to

$$\lim_{n\to\infty}\left\|x_{\tau(n)} - T\left(x_{\tau(n)}\right)\right\| = 0 \text{ and } \lim_{n\to\infty}\left\|x_{\tau(n)} - Q\left(x_{\tau(n)}\right)\right\| = 0. \tag{21}$$

Moreover, Lemma 3.3 and $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$ ($n \geq n_2$) guarantee that, for all $n \geq n_2$, $0 \leq \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + \alpha_{\tau(n)} M_{\tau(n)} < \alpha_{\tau(n)} M_{\tau(n)}$, and hence, $\mu\alpha_{\tau(n)}\|\nabla f(Q_\alpha(x_{\tau(n)}))\|^2 - \langle x_{\tau(n)} - x^\star, \nabla f(Q_\alpha(x_{\tau(n)}))\rangle > 0$ ($n \geq n_2$). Therefore, for all $n \geq n_2$,

$$\left\langle x_{\tau(n)} - x^\star, \nabla f\left(Q_\alpha\left(x_{\tau(n)}\right)\right)\right\rangle < \mu\alpha_{\tau(n)}\left\|\nabla f\left(Q_\alpha\left(x_{\tau(n)}\right)\right)\right\|^2. \tag{22}$$

A discussion in the same manner as in the proof of (19) leads us to

$$\begin{aligned} \left\langle x_{\tau(n)} - x^\star, \nabla f\left(x_{\tau(n)}\right)\right\rangle &\leq L(1-\alpha)\left\|x_{\tau(n)} - x^\star\right\|\left\|x_{\tau(n)} - Q\left(x_{\tau(n)}\right)\right\| \\ &\quad + \left\langle x_{\tau(n)} - x^\star, \nabla f\left(Q_\alpha\left(x_{\tau(n)}\right)\right)\right\rangle, \end{aligned}$$

which, together with (22), implies that

$$\left\langle x_{\tau(n)} - x^\star, \nabla f\left(x_{\tau(n)}\right)\right\rangle < L(1-\alpha)\left\|x_{\tau(n)} - x^\star\right\|\left\|x_{\tau(n)} - Q\left(x_{\tau(n)}\right)\right\| \\ + \mu\alpha_{\tau(n)}\left\|\nabla f\left(Q_\alpha\left(x_{\tau(n)}\right)\right)\right\|^2.$$

Thus, from (20), we find that, for all $n \geq n_2$,

$$\begin{aligned} c\Gamma_{\tau(n)} &\leq \left\langle x_{\tau(n)} - x^\star, \nabla f\left(x_{\tau(n)}\right)\right\rangle - \left\langle x_{\tau(n)} - x^\star, \nabla f(x^\star)\right\rangle \\ &< L(1-\alpha)\left\|x_{\tau(n)} - x^\star\right\|\left\|x_{\tau(n)} - Q\left(x_{\tau(n)}\right)\right\| + \left\langle x^\star - x_{\tau(n)}, \nabla f(x^\star)\right\rangle \\ &\quad + \mu\alpha_{\tau(n)}\left\|\nabla f\left(Q_\alpha\left(x_{\tau(n)}\right)\right)\right\|^2. \end{aligned}$$

Accordingly, (i) in Theorem 3.1, (21), and Lemma 3.2 lead us to

$$c\limsup_{n\to\infty}\Gamma_{\tau(n)} \leq \limsup_{n\to\infty}\left\langle x^\star - x_{\tau(n)}, \nabla f(x^\star)\right\rangle \leq 0,$$

which means that $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. Moreover, since Lemma 2.3 guarantees that $\lim_{n\to\infty}\Gamma_n = 0$, i.e., $(x_n)_{n\in\mathbb{N}}$ converges to $x^\star$. This completes the proof. □

4. Application of Algorithm 3.1 to Utility-based Bandwidth Allocation Problem with Operational Constraints

In this section, we apply Algorithm 3.1 with $T := T_{\mathrm{proj}}$ in (4), $Q := Q_{\mathrm{sp}}$ in (6), and $\alpha := 1/2$ to the utility-based allocation problem (Problem 1.1) on a simple network (Fig.1) that consists of two links and three sources. We will consider the case where the utility function is a *proportionally fair* (PF) function [23, 30, 36] defined by $\mathcal{U}_{\mathrm{pf}}(x_1, x_2, x_3) := \sum_{s=1}^3 w_s \log x_s$ ($(x_1, x_2, x_3)^T \in \mathbb{R}_+^3 \backslash \{0\}$) with $w_s := 1$ ($s = 1, 2, 3$). To see whether the first fixed point optimization algorithm, called the hybrid steepest descent method (HSDM) [40], converges or not, we can try to apply HSDM to Problem 1.1. HSDM is defined by $x_{n+1} := Q(x_n) + \alpha_n \nabla \mathcal{U}_{\mathrm{pf}}(Q(x_n))$ ($n \in \mathbb{N}$), where Q is quasi-nonexpansive and quasi-shrinking and $(\alpha_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Meanwhile, $(z_n)_{n\in\mathbb{N}}$ defined by $z_{n+1} := \lambda_n z_n + (1-\lambda_n)T_1(T_2(z_n))$ ($n \in \mathbb{N}$) converges to a point in $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2)$, where $(\lambda_n)_{n\in\mathbb{N}} \subset [0,1]$ satisfies $\sum_{n=1}^\infty \lambda_n(1-\lambda_n) = \infty$, $z_1 \in \mathbb{R}^S$, and T_1 and T_2 are nonexpansive with $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) \neq \emptyset$ [4, Corollary 4.37, Theorem 5.14]. Since $C \cap C_{\mathrm{op}} = \mathrm{Fix}(T_{\mathrm{proj}}) \cap \mathrm{Fix}(Q_{\mathrm{sp}}) \neq \emptyset$, we can use $T_{\mathrm{proj}}Q_{\mathrm{sp}}$ to find a point in $\mathrm{Fix}(T_{\mathrm{proj}}) \cap \mathrm{Fix}(Q_{\mathrm{sp}})$. Hence, we can replace HSDM for optimization problems over $\mathrm{Fix}(Q_{\mathrm{sp}})$ by

$$x_{n+1} := T_{\mathrm{proj}}(Q_{\mathrm{sp}}(x_n)) + \alpha_n \nabla \mathcal{U}_{\mathrm{pf}}(T_{\mathrm{proj}}(Q_{\mathrm{sp}}(x_n))) \tag{23}$$

to enable us to consider optimization problems over $\mathrm{Fix}(T_{\mathrm{proj}}) \cap \mathrm{Fix}(Q_{\mathrm{sp}})$. From the above discussion, we can expect that HSDM defined by (23) converges to a point in $\mathrm{Fix}(T_{\mathrm{proj}}) \cap \mathrm{Fix}(Q_{\mathrm{sp}})$ and solves Problem 1.1. We chose five random initial points and executed Algorithm 3.1 and HSDM defined by (23) for chosen initial points. The following graphs plot the mean values of the fifth execution. The computer used in the experiment had an Intel Boxed Core i7 i7-870 2.93 GHz 8M CPU and 8 GB of memory. The language was MATLAB 7.13. In the experiments, we used $\mu := 1/10^2$, $\alpha_n := 1/(n+1)^{0.5}$, and $\beta_n := 1/2$ $(n \in \mathbb{N})$ in Algorithm 3.1. Theorem 3.1 guarantees that Algorithm 3.1 converges to a unique solution to Problem 1.1.

To check whether Algorithm 3.1 and HSDM defined by (23) converge in the constraint set, $C \cap C_{\mathrm{op}} = \mathrm{Fix}(T_{\mathrm{proj}}) \cap \mathrm{Fix}(Q_{\mathrm{sp}})$, in Problem 1.1, we employed the following evaluation function:

$$\begin{aligned} D_n &:= \|x_n - T_{\mathrm{proj}}(x_n)\| + \|x_n - Q_{\mathrm{sp}}(x_n)\| \\ &= \left\| x_n - P_{\mathbb{R}^3_+}\left(P_{C_1}\left(P_{C_2}(x_n)\right)\right)\right\| + \begin{cases} \dfrac{|\mathcal{P}(x_n) - p|}{\|\mathcal{P}'(x_n)\|} & \text{if } \mathcal{P}(x_n) > p, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $x_n \in \mathbb{R}^3$ is the nth approximation to the solution. Since any x with $\|x - T_{\mathrm{proj}}(x)\| = 0$ (resp. $\|x - Q_{\mathrm{sp}}(x)\| = 0$) satisfies $x \in \mathrm{Fix}(T_{\mathrm{proj}}) = C := \mathbb{R}^3_+ \cap C_1 \cap C_2$ (resp. $x \in \mathrm{Fix}(Q_{\mathrm{sp}}) = C_{\mathrm{op}} := \{x \in \mathbb{R}^3 \colon \mathcal{P}(x) \leq p\}$), the convergence of $(D_n)_{n\in\mathbb{N}}$s to 0 implies that the algorithms converge in $C \cap C_{\mathrm{op}}$. In the experiment, we used $\mathcal{P}(x) := \mathcal{P}(x_1)$ $(x_1 \in \mathbb{R})$ defined by $2x_1 - (7/2)x^0$ $(x_1 \geq (3/2)x^0)$, or $x_1 - 2x^0$ $(x^0 < x_1 < (3/2)x^0)$, or $-x^0$ $(x_1 \leq x^0)$, $x^0 := 0.1$, and $p := 2$. We can verify that the optimal solution maximizing $\mathcal{U}_{\mathrm{pf}}(x) := \sum_{s=1}^{3} \log x_s$ over C is $x^* \approx (3.8, 1.8, 1.2)^T$. By using $C_{\mathrm{op}} := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \colon \mathcal{P}(x_1) \leq p\}$, the operator can decrease the optimal transmission rate of source 1 from $x_1^* \approx 3.8$ to $(7/4)x^0 + (1/2)p$ $(= 1.175)$, and hence, the transmission rate of source 3 becomes larger than the previous $x_3^* \approx 1.2$. Therefore, the operational policy using $\mathcal{P}(x_1)$ is that the operator tries to decrease the transmission rate of source 1 sharing link 1 in order to increase the transmission rate of source 3 sharing link 1.

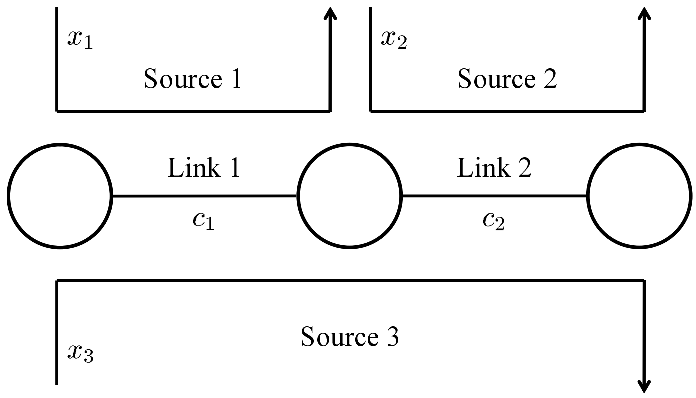

Figure 1: Network with two links and three sources with $c_1 := 5$, $c_2 := 3$ ($C_l := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \colon x_l + x_3 \leq c_l\}$ $(l = 1, 2)$).

Figure 2 plots the behaviors of $D_n := \|x_n - T_{\mathrm{proj}}(x_n)\| + \|x_n - Q_{\mathrm{sp}}(x_n)\|$ $(n = 1, 2, \ldots, 100)$ for Algorithm 3.1 (Proposed) and HSDM defined by (23). Note that, if the $(D_n)_{n\in\mathbb{N}}$s converge to 0, the algorithms converge in the constraint set in Problem 1.1. From this figure, we can see that Algorithm 3.1 quickly diminishes the value of D_n and converges in $C \cap C_{\mathrm{op}}$, whereas HSDM diminishes D_n slowly and the D_{100} it generates is about 10^{-1}.

Figure 3 shows the behaviors of $\mathcal{U}_{\mathrm{pf}}(x_n)$ $(n = 1, 2, \ldots, 100)$ and presents the required iterations of Algorithm 3.1. This figure indicates that Algorithm 3.1 is stable for $n \geq 20$. Figures 2 and 3 show that Algorithm 3.1 converges and finds the unique solution to Problem 1.1 ($x_1^\star \approx 1.175, x_2^\star \approx 1.500, x_3^\star \approx 1.500$), as promised by Theorem 3.1. Meanwhile, these figures show that HSDM does not converge to the solution because it does not converge in $C \cap C_{\mathrm{op}}$, as seen in Figure 2.

Now, let us compare (i) the PF allocation for all sources under capacity constraints and an operational constraint with (ii) the PF allocation for all sources under only the capacity constraints. The allocation for (i) is $(x_1^\star, x_2^\star, x_3^\star) \approx (1.175, 1.500, 1.500)$, and the allocation for (ii) is $(x_1^*, x_2^*, x_3^*) \approx (3.800, 1.800, 1.200)$. By using the operational constraint set,

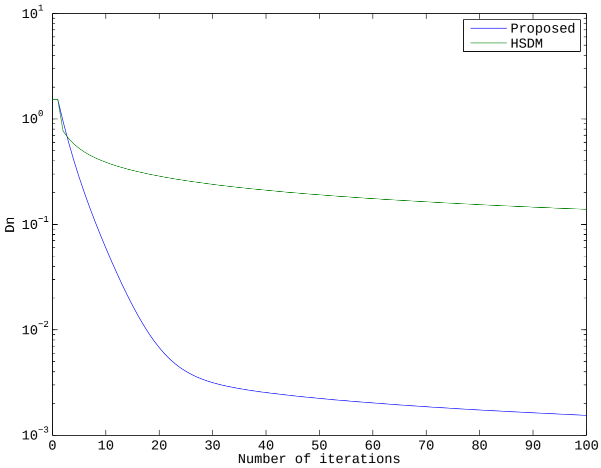

Figure 2: Behavior of $D_n := \|x_n - T_{\mathrm{proj}}(x_n)\| + \|x_n - Q_{\mathrm{sp}}(x_n)\|$ $(n = 1, 2, \ldots, 100)$.

$C_{\mathrm{op}} := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \colon \mathcal{P}(x_1) \leq p\}$, to limit the transmission rate of source 1, the operator can decrease the transmission rate of source 1 from $x_1^* \approx 3.800$ to $x_1^\star \approx 1.175$ and increase the transmission rate of source 3 from $x_3^* \approx 1.200$ to $x_3^\star \approx 1.500$. This is because sources 1 and 3 share the same link 1 and have the same utility function $\mathcal{U}_1(x) = \mathcal{U}_3(x) := \log x$ $(x \in \mathbb{R}_+ \setminus \{0\})$, and decreasing the transmission rate of source 1 results in an increase in the transmission rate of source 3. Source 3 cannot send data by $c_1 - x_1^\star \approx 3.825$ because it shares link 2 with capacity $c_2 := 3$. Since sources 2 and 3 share link 2 and have the same utility function $\mathcal{U}_2(x) = \mathcal{U}_3(x) := \log x$ $(x \in \mathbb{R}_+ \setminus \{0\})$, the optimal transmission rates of sources 2 and 3 satisfy $x_2^\star = x_3^\star$ and $x_2^\star + x_3^\star = c_2$, i.e., $x_2^\star, x_3^\star = 1.500$.

Algorithm 3.1 with a slowly diminishing sequence can solve the network bandwidth allocation problem with a concave utility function, while HSDM does not always converge in the constraint set of the network bandwidth allo-

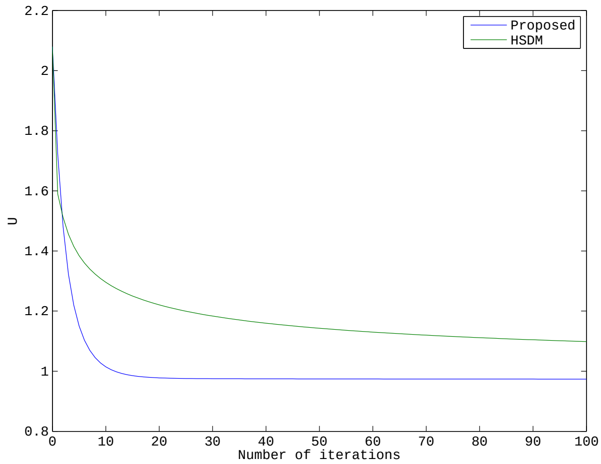

Figure 3: Behavior of $\mathcal{U}_{\mathrm{pf}}(x) := \sum_{s=1}^{3} \log x_s$.

cation problem. The above observations suggest that the proposed algorithm is a more efficient way of solving network bandwidth allocation problems under capacity constraints and operational constraints in comparison with existing algorithms such as HSDM.

5. Conclusion

This paper discussed the variational inequality problem over the intersection of the fixed point sets of a nonexpansive mapping and quasi-nonexpansive mapping, including the network bandwidth allocation problem under capacity and operational constraints. To solve the problem, we devised a fixed point optimization algorithm based on iterative techniques for optimization over the fixed point sets of nonexpansive mappings and presented its convergence analysis. We applied the algorithm to the network bandwidth allocation problem and compared it with an existing algorithm. The numerical comparisons showed that there is a possibility that the existing algorithm does not work for the network bandwidth allocation problems and suggested that the proposed algorithm is an efficient way to achieve the optimal bandwidth allocation.

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